2021

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 6

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Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

Model Solution	Scoring	
(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.		
e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions	**1 point**
To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x}\,dx$ is finite (converges).	Improper integral	**1 point**
$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big\vert_0^b\right) = \lim_{b\to\infty} \left(-e^{-b} + e^0\right) = 1$ Because the integral $\int_0^{\infty} e^{-x}\,dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation	**1 point**

Scoring notes:

- To earn the first point a response must list all three conditions: e^{-x} is positive, decreasing, and continuous.
- The second point is earned for correctly writing the improper integral or for presenting a correct limit equivalent to the improper integral (for example, $\lim_{b\to\infty}\int_0^b e^{-x}\,dx$).
- To earn the third point a response must correctly use limit notation to evaluate the improper integral, find an evaluation of e^0 (or 1), and conclude that the integral converges or that the series converges.
- If an incorrect lower limit of 1 is used in the improper integral, then the second point is not earned. In this case, if the correct limit ($1/e$) is presented, then the response is eligible for the third point.
- If the response only relies on using a geometric series approach, then no points are earned [0-0-0].
- A response that presents an evaluation with ∞, such as $e^{-\infty} = 0$, does not earn the third point.

Total for part (a) 3 points

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

$\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left\lvert\frac{(-1)^n}{2e^n + 3}\right\rvert} = \lim_{n\to\infty} \frac{2e^n + 3}{e^n} = 2$ The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \left\lvert\frac{(-1)^n}{2e^n + 3}\right\rvert$ converges by the limit comparison test. Thus, the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.	Sets up limit comparison	**1 point**
	Explanation	**1 point**

Scoring notes:

- The first point is earned for setting up the limit comparison, with or without absolute values. Limit notation is required to earn this point.
- The reciprocal of the given ratio is an acceptable alternative; the limit in this case is $1/2$.
- The second point cannot be earned without the use of absolute value symbols, which can occur explicitly or implicitly (e.g., a response might set up the limit comparison initially as $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\frac{1}{2e^n + 3}}$).
- Earning the second point requires correctly evaluating the limit and noting that the limit is a positive number. For example, $L = 2 > 0$ or $L = 1/2 > 0$. Therefore, comparing the limit L to 1 does not earn the explanation point.
- A response does not have to repeat that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.
- A response that draws a conclusion based only on the sequence (such as $\frac{1}{e^n}$) without referencing a series does not earn the second point.
- If the response does not explicitly use the limit comparison test, then no points are earned in this part.
- A response cannot earn the second point for just concluding that "the series" converges absolutely because there are multiple series in this part of the problem. The response must specify that the series $g(1)$ or $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Total for part (b) 2 points

(c) Determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \dfrac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \dfrac{2e^n+3}{(-1)^n x^n} \right\rvert = \left\lvert \dfrac{(2e^n+3)x^{n+1}}{(2e^{n+1}+3)x^n} \right\rvert = \dfrac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert$	Sets up ratio	**1 point**
$\lim\limits_{n\to\infty} \dfrac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert = \dfrac{1}{e}\lvert x\rvert$	Computes limit of ratio	**1 point**
$\dfrac{1}{e}\lvert x\rvert < 1 \Rightarrow \lvert x\rvert < e$ The radius of convergence is $R = e$.	Answer	**1 point**

Scoring notes:

- The first point is earned for $\dfrac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \dfrac{2e^n+3}{(-1)^n x^n}$ or the equivalent. Once earned, this point cannot be lost.
- The second point cannot be earned without the first point.
- To be eligible for the third point the response must have found a limit for a presented ratio such that the limiting value of the coefficient on $\lvert x\rvert$ is finite and not 0. The third point is earned for setting up an inequality such that the limit is less than 1, solving for $\lvert x\rvert$, and interpreting the result to find the radius of convergence.
- The radius of convergence must be explicitly presented, for example, $R = e$. The third point cannot be earned by presenting an interval, for example $-e < x < e$, with no identification of the radius of convergence.

Total for part (c) 3 points

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ decrease in magnitude to 0. The alternating series error bound for the error of the approximation is the absolute value of the third term of the series. $\text{Error} \le \left\lvert \frac{(-1)^2}{2e^2 + 3} \right\rvert = \frac{1}{2e^2 + 3}$	Answer	**1 point**

Scoring notes:

- A response of $\frac{1}{2e^2 + 3}$ earns this point.

Total for part (d) 1 point

Total for question 6 9 points

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

The function $\frac{1}{e^n}$ is continuous, positive, and decreasing for $n \geq 0$

$\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by integral test

$$\cancel{\lim_{b\to\infty} \int_0^b \frac{1}{e^b}\,db} = \lim_{b\to\infty} \int_0^b \frac{1}{e^n}\,dn = \lim_{b\to\infty} \int_0^b e^{-n}\,dn$$

$$= \lim_{b\to\infty} \left[-e^{-n}\right]_0^b = \lim_{b\to\infty} \left[-e^{-b} - (-e^0)\right]$$

$$= 0 + 1 = 1$$

Response for question 6(b)

$$g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$$

compare to $\sum_{n=0}^{\infty} \frac{1}{e^n}$ geometric $r = \frac{1}{e} < 1$ converges

absolute value of $g(1)$

$$\lim_{n\to\infty} \frac{\frac{1}{2e^n+3}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{e^n}{2e^n+3} = \frac{1}{2} > 0$$

$\frac{1}{2e^n+3} > 0$

$\frac{1}{e^n} > 0$

$g(1)$ converges absolutely by the Limit Comparison Test

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Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n+3}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{x^n} \right| = \left| \frac{x}{e} \right|$$

$$\left| \frac{x}{e} \right| < 1$$

$$|x| < e$$

$$R = e$$

radius of convergence for g

Response for question 6(d)

approximation using first two terms

Error $\leq$ | third term of the series $g(1)$ |

$$\text{Error} \leq \left| \frac{1}{2e^2+3} \right|$$

$$\text{Error} \leq \frac{1}{2e^2+3}$$

Answer QUESTION 6 parts (a) and (b) on this page.

$e^{-x} = -e^{-x}$

Response for question 6(a)

$\sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{1}{e^x}$ * function [illegible] equal to series [illegible]
* series can be approximated by function

$\int_0^{\infty} \frac{1}{e^x} = \lim_{b\to\infty} \int_0^b \frac{1}{e^x} = \lim_{b\to\infty} -\frac{1}{e^x}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{e^b} + \frac{1}{e^0} = 1$

Since $\int_0^{\infty} \frac{1}{e^x} = 1$, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by the integral test.

Response for question 6(b)

$\sum_{n=0}^{\infty} \frac{1}{e^n} > \sum_{n=0}^{\infty} \frac{1}{2e^n+3}$

$\lim_{n\to\infty} \left| \frac{e^n}{2e^n+3} \right| = \lim_{n\to\infty} \left| \frac{e^n}{2e^n} \right| = \frac{1}{2} < 1$

So, converges absolutely by limit comparison test.

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{n+1}}{(-1)^{n}\cdot x^{n}}\cdot\frac{2e^{n}+3}{2e^{n+1}+3}\right| = \lim_{n\to\infty}\left|(-1)(x)\cdot\frac{1}{2e}\right|$$

$$-1 < -\frac{x}{2e} < 1$$

$$2e > x > -2e$$

$\boxed{\text{radius of convergence} = 2e}$

Response for question 6(d)

*since first two terms are used $n=0$ and $n=1$ for approximation, third term will give the upper bound on error of approximation

$$\text{error} \leq \frac{(-1)^2}{2e^2+3} \rightarrow n=2$$

$$\boxed{\text{error} \leq \frac{1}{2e^2+3}}$$

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

Conditions are positive, continuous, and decreasing.

$$\int_0^\infty e^{-n}\,dn = -e^{-n}\Big|_0^\infty = \frac{-1}{e^n}\Big|_0^\infty = \left[0 - \frac{-1}{e^0}\right] = \frac{1}{1} = 1$$

Since the integral went to a finite value in 1, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (~~It~~ The integral test does not say where it converges to though)

Response for question 6(b)

$$\lim_{n\to\infty} \frac{\frac{(-1)^n}{2e^n+3}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{(-1)^n e^n}{2e^n+3} \to \lim_{n\to\infty} \left|\frac{e^n}{2e^n+3}\right| = \frac{1}{2}$$

The limit test converges to less than 1, so it converges there absolutely and thus there is no need to check conditional convergence.

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Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3}\cdot\frac{2e^{n}+3}{(-1)^{n}x^{n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)x+3}{2e+3}\right| \to \frac{-x+3}{2e+3}<1$$

$$-x+3<2e+3$$

$$-x<2e$$

$$x>2e$$

$$3<x<2e+3$$

Response for question 6(d)

Need 3rd term

$$\frac{-1}{2e^{3}+3}$$

Upper bound on error of the approximation can only be as big as the next term, thus the maximum error is $-\dfrac{1}{2e^{3}+3}$

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the function g has derivatives of all orders for all real numbers and students were given the Maclaurin series for g.

In part (a) students were asked to state the conditions necessary to use the integral test to determine whether the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, and then to use the integral test to show that the series converges. A correct response should state that the integral test requires the function $\frac{1}{e^x}$ to be positive, decreasing, and continuous on the interval $[0, \infty)$. The response should continue by demonstrating that the improper integral $\int_0^{\infty} e^{-x}\,dx$ is finite and therefore converges, so $\sum_{n=0}^{\infty} e^{-n}$ also converges.

In part (b) students were asked to use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely. A correct response should use correct notation to show that $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left|\frac{(-1)^n}{2e^n + 3}\right|}$ is finite and positive and reference the convergence of $\sum_{n=0}^{\infty} \frac{1}{e^n}$ determined in part (a).

In part (c) students were asked to determine the radius of convergence of the Maclaurin series for g. A correct response should use the ratio test to determine the radius of convergence is $R = e$.

In part (d) students were asked to use the alternating series error bound to determine an upper bound on the error when the first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. A correct response should indicate that the approximate error is bounded by the absolute value of the third term of the series, $\frac{1}{2e^2 + 3}$.

Sample: 6A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the response earned the first point in the first line by stating that "the function $\frac{1}{e^n}$ is continuous, positive, and decreasing." The response earned the second point in the fourth line by presenting the limit $\lim_{b\to\infty} \int_0^b \frac{1}{e^n}\,dn$. The response earned the third point by presenting a correct antiderivative on the fifth line, correctly evaluating the limit on the sixth line, and stating a correct conclusion on the third line. In part (b) the response earned the first point

Question 6 (continued)

in the second line by presenting the limit (with or without absolute values) $\lim_{n\to\infty} \frac{\frac{1}{2e^n + 3}}{\frac{1}{e^n}}$. The response earned the second point by correctly evaluating the limit resulting in $\frac{1}{2}$, referencing that $\frac{1}{2} > 0$, and presenting a correct conclusion in the lower right. In part (c) the response earned the first point in the second line by presenting the ratio (with or without a limit) $\left|\frac{x^{n+1}}{2e^{n+1} + 3} \cdot \frac{2e^n + 3}{x^n}\right|$. In the absence of the factors $(-1)^{n+1}$ and $(-1)^n$, the response must have absolute values to earn this point. The response earned the second point on the second line by correctly evaluating the limit. The response earned the third point on the fifth line by correctly identifying the radius of convergence. In part (d) the response earned the point by providing the correct answer.

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the response did not earn the first point because the response does not identify the three needed conditions for the function $\frac{1}{e^x}$. The response earned the second point in the second line by presenting the improper integral (with or without a differential) $\int_0^\infty \frac{1}{e^x}$. The response earned the third point by presenting a correct antiderivative, correctly evaluating the limit, and stating a correct conclusion. In part (b) the response earned the first point in the second line by presenting the limit (with or without absolute values) $\lim_{n\to\infty}\left|\frac{e^n}{2e^n + 3}\right|$. The response did not earn the second point because the response compares $\frac{1}{2}$ with 1 and not 0. In part (c) the response earned the first point in the first line by presenting the ratio (with or without a limit or absolute values) $\frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \cdot \frac{2e^n + 3}{2e^{n+1} + 3}$. The response did not earn the second point because the response, by bounding $\frac{-x}{2e}$ between both -1 and 1, implies that their limit is $\frac{-x}{2e}$, which is incorrect. The response is eligible for the third point because the response presents a value for the limit and considers an absolute value by presenting $-1 < \frac{-x}{2e} < 1$. The response earned the third point by presenting the radius of convergence $2e$, which is consistent with their limit. In part (d) the response earned the point by providing the correct answer.

Question 6 (continued)

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the response did not earn the first point because the response states the correct conditions but does not reference the function $\frac{1}{e^x}$. The response earned the second point in the second line by presenting the improper integral $\int_0^\infty e^{-n}\,dn$. The response did not earn the third point because on the second line, the response applies the Fundamental Theorem of Calculus to an improper integral. In part (b) the response earned the first point on the first line by presenting the limit (with or without absolute values) $\lim_{n\to\infty} \frac{\frac{(-1)^n}{2e^n+3}}{\frac{1}{e^n}}$. The response did not earn the second point because in the second line, the response references a limit "less than 1" whereas the correct application of the limit comparison test would require referencing a limit greater than 0. In part (c) the response earned the first point on the first line by presenting the ratio (with or without a limit or absolute values) $\frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3}\cdot\frac{2e^n+3}{(-1)^n x^n}$. The response did not earn the second point because the evaluation of the limit is incorrect. The response did not earn the third point because the response does not identify a radius of convergence consistent with their limit evaluation. In part (d) the response did not earn the point as the response presents the incorrect answer.