Chief Reader Report on Student Responses:

2019 AP® Calculus AB and Calculus BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC):	1,165		
Calculus AB			
• Number of Students Scored	300,659		
• Score Distribution	Exam Score	N	%At
	5	57,352	19.1
	4	56,206	18.7
	3	61,950	20.6
	2	70,088	23.3
	1	55,063	18.3
• Global Mean	2.97		
Calculus BC			
• Number of Students Scored	139,195		
• Score Distribution	Exam Score	N	%At
	5	59,797	43.0
	4	25,720	18.5
	3	27,166	19.5
	2	19,303	13.9
	1	7,209	5.2
• Global Mean	3.80		
Calculus BC Calculus AB Subscore			
• Number of Students Scored	139,195		
• Score Distribution	Exam Score	N	%At
	5	68,859	49.5
	4	32,740	23.5
	3	18,344	13.2
	2	13,433	9.7
	1	5,819	4.2
• Global Mean	4.04		

The following comments on the 2019 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Stephen Davis of Davidson College. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1 **Topic:** Modeling Rates

Max. Points: 9 **Mean Score: AB1:** 3.70; **BC1:** 5.14

What were the responses to this question expected to demonstrate?

In this problem, fish enter and leave a lake at rates modeled by functions E and L given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$ and $L(t) = 4 + 2^{0.1t^2}$, respectively. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight $(t = 0)$.

In part (a) students were asked to find the number of fish entering the lake between midnight $(t = 0)$ and 5 A.M. $(t = 5)$ and to provide the answer rounded to the nearest whole number. A response should demonstrate an understanding that a definite integral of the rate at which fish enter the lake over the time interval $0 \le t \le 5$ gives the number of fish that enter the lake during that time period. The numerical value of the integral $\int_0^5 E(t)\,dt$ should be obtained using a graphing calculator.

In part (b) students were asked for the average number of fish that leave the lake per hour over the 5-hour period $0 \le t \le 5$. A response should demonstrate that "number of fish per hour" is a rate, so the question is asking for the average value of $L(t)$ across the interval $0 \le t \le 5$, found by dividing the definite integral of L across the interval by the width of the interval. The numerical value of the expression $\frac{1}{5}\int_0^5 L(t)\,dt$ should be obtained using a graphing calculator.

In part (c) students were asked to find, with justification, the time t in the interval $0 \le t \le 8$ when the population of fish in the lake is greatest. The key understanding here is that the rate of change of the number of fish in the lake, in number of fish per hour, is given by the difference $E(t) - L(t)$. Analysis of this difference using a graphing calculator shows that, for $0 \le t \le 8$, the difference has exactly one sign change, occurring at $t = 6.20356$. Before this time, $E(t) - L(t) > 0$, so the number of fish in the lake is increasing; after this time, $E(t) - L(t) < 0$, so the number of fish in the lake is decreasing. Thus the number of fish in the lake is greatest at $t = 6.204$ (or 6.203). An alternative justification uses the definite integral of $E(t) - L(t)$ over an interval starting at $t = 0$ to find the net change in the number of fish in the lake from time $t = 0$. The candidates for when the fish population is greatest are the endpoints of the time interval $0 \le t \le 8$ and the one time when $E(t) - L(t) = 0$, namely $t = 6.20356$. Numerical evaluation of the appropriate definite integrals on a graphing calculator shows that the number of fish in the lake is greatest at $t = 6.204$ (or 6.203).

In part (d) students were asked whether the rate of change in the number of fish in the lake is increasing or decreasing at time $t = 5$. A response should again demonstrate the understanding that the rate of change of the number of fish in the lake is given by the difference $E(t) - L(t)$, and whether this rate is increasing or decreasing at time $t = 5$ can be determined by the sign of the derivative of the difference at that time. Using a graphing calculator to find that $E'(5) - L'(5) < 0$ leads to the conclusion that the rate of change in the number of fish in the lake is decreasing at time $t = 5$.

For part (a) see LO CHA-4.E/EK CHA-4.E.1, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1. For part (c) see LO FUN-4.B/EK FUN-4.B.1. For part (d) see LO CHA-3.C/EK CHA-3.C.1, LO CHA-2.D/EK CHA-2.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) a clear majority of responses showed understanding of a need to integrate a rate of change in the number of fish to find a number of fish. Most responses opted correctly for $\int_0^5 E(t)\,dt.$ Some responses, however, overlooked the key word "enter" and used $E(t) - L(t)$ as the integrand, instead. Some responses presented the rate $E(5)$ as an answer. There were relatively few issues obtaining a numerical value from a graphing calculator.

In part (b) a clear majority of responses again showed understanding of a need to integrate a rate of change, and most opted correctly for $\int_0^5 L(t)\,dt,$ earning the first point. Some responses failed to divide by 5 to obtain the average number of fish per hour leaving the lake. Among those responses that had a correct setup of $\frac{1}{5}\int_0^5 L(t)\,dt,$ some had rounding errors that resulted in a final answer that was not accurate to three decimal places, and others (perhaps influenced by the "nearest whole number" instruction of part (a)) presented an integer answer. Responses with these errors did not earn the second point in this part.

In part (c) many responses showed understanding of the need to determine a sign change in the difference of rates, $E(t) - L(t),$ and most of these responses solved for $E(t) = L(t)$ successfully. Some responses arrived at the solution $t = 6.204$ without identifying the equation being solved. Some incorrect responses maximized either the rate $E(t)$ or the rate $E(t) - L(t),$ earning no points in this part. Some responses tried justifications based upon the sign of $E(t) - L(t)$ (see the first of the two solutions presented in the scoring guidelines), but failed to be complete enough to bridge the gap between justifying a local maximum at $t = 6.204$ and justifying a maximum across the entire interval $0 \le t \le 8.$

In part (d) many responses conveyed a consideration of $E'(5)$ and $L'(5)$ to earn the first point. Many of these responses were able to include a complete explanation in support of a decreasing rate of change in the number of fish, but some failed to directly compare $E'(5)$ to $L'(5)$ and give a full explanation of a "decreasing" answer. Some responses included errors in the numerical value of one of $E'(5)$ or $L'(5),$ and so earned the first point but were not eligible for the second point. Some other responses included extraneous or irrelevant information about $E(5)$ versus $L(5)$ as a "necessary" part of the explanation, and so also did not earn the second point.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) computing the net change in the number of fish as in $\int_0^5 E(t)\,dt - \int_0^5 L(t)\,dt = 123.163$ 123 fish enter the lake from midnight to 5 a.m.	$\int_0^5 E(t)\,dt = 153.458$ 153 fish enter the lake from midnight to 5 a.m.
In part (b) presenting the average rate that fish leave the lake rounded to the nearest whole number as in $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059$ The average number of fish that leave the lake per hour from midnight to 5 a.m. is 6 fish per hour.	$\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059$ The average number of fish that leave the lake per hour from midnight to 5 a.m. is 6.059 fish per hour.

In part (c) using a local argument to "justify" a global maximum: The number of fish in the lake is greatest at $t = 6.204$ because $E(t) - L(t)$ changes sign from positive to negative there.	The number of fish in the lake is greatest at $t = 6.204$ because $E(t) - L(t) > 0$ for $0 \le t < 6.20356$ and $E(t) - L(t) < 0$ for $6.20356 < t \le 8$.
In part (d) basing an answer on just one of the functions E or L as in $L(0) = 5;\ L(5) = 9.657$ Therefore decreasing because more fish are leaving at 5 A.M. than at midnight.	$E'(5) - L'(5) = -10.723 < 0$ Because $E'(5) - L'(5) < 0,$ the rate of change in the number of fish is decreasing at time $t = 5$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Students often have difficulty interpreting and dealing with functions that describe rates of change, as is the case with the functions E and L in this problem. Perhaps they read the problem too quickly and interpret the functions as measuring a population (the number of fish entering or leaving the lake), versus measuring a rate of change in population (the number of fish per hour entering or leaving the lake). Or perhaps some students jump right to the goal (find a maximum; determine increasing or decreasing) and grab the most convenient function expressions, $E(t)$ and/or $L(t)$, without regard to the context of what these functions actually describe, resulting in time spent on work that will not be rewarded with points. Practice in careful reading and interpretation may be able to help some of these students. Also, the graphing calculator was a necessary tool for this problem: evaluating definite integrals in parts (a) and (b); solving an equation in part (c); and evaluating numerical derivatives in part (d). Some students, however, graphed the functions E and L and based their answers upon characteristics of these graphs as they appeared on their calculator screens, which by itself does not provide the basis for a justification or explanation sufficient for credit. Teachers can emphasize going beyond the evidence that such graphs provide to a calculus-based justification. Further, some responses included justifications that were framed in less-than-precise prose when a succinct mathematical equation or expression would more safely convey the needed information. Teachers can highlight the power and clarity of symbolic expressions that incorporate correct notation. Finally, a few responses showed evidence of calculators set in degree mode when dealing with the function E. Teachers can continue to remind their students to put their calculators in radian mode on AP Exam day before beginning the AP Calculus Exam.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- In parts (a) and (b), students needed to identify and present appropriate definite integrals (Skills 1.D and 4.C), correctly compute values using a graphing calculator (Skill 1.E), and present answers using appropriate rounding (Skill 4.E). For additional practice with the content and skills in parts (a) and (b), teachers may want to assign Topic Questions from Topics 8.3 and 8.1, or use the Personal Progress Checks for additional review of Unit 8 in the *AP Calculus AB and BC Course and Exam Description* (*CED*). Teachers might also want to search the Question Bank for items associated with these topics or skills.
- In part (c) students needed to justify a conclusion about the absolute maximum value of a function over an interval. The instructional activity "Create a Plan" in Unit 5 of the *CED* (p. 95) is designed to help students organize their thinking as they prepare to justify an absolute maximum value. Skills leading to a complete justification may include 2.A (identifying the structure of the optimization question), 3.B (identifying the First Derivative Test as appropriate to justify an absolute maximum on an interval, provided there is only one critical value on the interval), 3.C (confirming that there is only one critical value on the interval), 3.D (applying the test

to draw the conclusion), and 3.E (providing reasons or rationales for the conclusion). The Question Bank can be searched to identify relevant questions by these skills. Topic Questions from Topics 5.2–5.7, 5.10, and 5.11 offer extra practice for students with the content and skills in part (c) and the Personal Progress Checks for Unit 5 offer a good opportunity for review.

- In part (d) students needed to determine whether a rate is increasing or decreasing and explain their reasoning. The instructional activity suggested for Topic 5.3 on p. 95 of the *CED*, "Critique Reasoning," helps students to improve their own reasoning and explanations.

Question AB2 **Topic:** Particle Motion

Max. Points: 9 **Mean Score:** 2.83

What were the responses to this question expected to demonstrate?

In this problem a particle P moves along the x-axis with velocity given by a differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. The particle starts at the origin at time $t = 0$, and selected values of $v_P(t)$ are given in a table.

In part (a) students were asked to justify why there is at least one time t, for $0.3 \le t \le 2.8$, when the acceleration of particle P is 0. A response should demonstrate that the hypotheses of the Mean Value Theorem are satisfied on the given interval and that applying the Mean Value Theorem to v_P on $[0.3, 2.8]$ leads to the desired conclusion.

In part (b) students were asked to approximate $\int_0^{2.8} v_P(t)\,dt$ using a trapezoidal sum and data from the table of selected values of $v_P(t)$. A response should demonstrate the form of a trapezoidal sum using the three subintervals indicated.

In part (c) a second particle, Q, is introduced, also moving along the x-axis, and with velocity $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Students were asked to find the time interval during which $v_Q(t) \ge 60$ and to find the distance traveled by particle Q during this time interval. Using a graphing calculator to find the interval, a response should demonstrate that the distance traveled by particle Q is given by the definite integral of the absolute value of v_Q over this time interval. The value of this integral is found using the numerical integration capability of a graphing calculator.

In part (d) students were given that particle Q starts at position $x = -90$ at time $t = 0$ and were asked to use the approximation from part (b) and the velocity function v_Q introduced in part (c) to approximate the distance between particles P and Q at time $t = 2.8$. A response should demonstrate that the integral approximated in part (b) gives the position of particle P at time $t = 2.8$, and that the position of particle Q at this time is found by adding the particle's initial position, $x = -90$, to $\int_0^{2.8} v_Q(t)\,dt$. The student's response should report the difference between these two positions.

For part (a) see LO CHA-2.A/EK CHA-2.A.1, LO FUN-1.B/EK FUN-1.B.1. For part (b) see LO LIM-5.A/EK LIM-5.A.2. For part (c) see LO CHA-4.C/EK CHA-4.C.1, LO LIM-5.A/ EK LIM-5.A.3. For part (d) see LO CHA-4.C/EK CHA-4.C.1, LO LIM-5.A/EK LIM-5.A.3. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) many responses showed understanding that the Mean Value Theorem is a relevant tool. However, many of these responses failed to present an appropriate difference quotient and responses often omitted the requisite condition that v_P is continuous on the interval. Some responses attempted an argument that $v_P{}'(t)$ must change sign on the interval $(0.3, 2.8)$ (two applications of the Mean Value Theorem could justify this), and then appealed to the Intermediate Value Theorem applied to $v_P{}'$. However, it is not given that $v_P{}'$ is continuous, so the conditions of the Intermediate Value Theorem are not necessarily satisfied. These responses could earn the first point but not the second. Some responses surmised incorrectly from the table that v_P must decrease for $0.3 < t < 1.7$ and increase for $1.7 < t < 2.8,$ concluding that $v_P'(1.7) = 0.$ These responses earned no points in this part.

In part (b) many responses included a correct trapezoidal sum or correctly computed this sum as the average of left and right Riemann sums. However, a significant portion of these responses made arithmetic errors in simplifying, and thus missed the opportunity to earn the point for this part. Some responses computed an approximation but failed to communicate the calculations upon which the approximation was based.

In part (c) many responses presented a correct interval for $v_Q(t) \ge 60,$ although some only solved for $v_Q(t) = 60$ and omitted an explicit declaration of the requested interval. Many responses showed understanding that distance traveled is computed by a definite integral of the absolute value of velocity. (Here $v_Q(t) \ge 60,$ so "absolute value" is not required.) Some responses presented the interval, integral, and distance traveled, including correct decimal presentations, earning all 3 points; others had decimal presentation errors or used intermediate rounding, leading to a distance value that was not accurate to three decimal places.

In part (d) many responses showed understanding that the position of particle Q is found using an integral of $v_Q(t).$ Some responses used $\int_0^{2.8} v_Q(t)\,dt$ as the position of particle Q at time $t = 2.8,$ missing the nuance that this integral is the net change in position of the particle across the interval $0 \le t \le 2.8$ and should be added to the initial position, $x = -90.$

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) a reference to v_P must be unambiguous to earn the first point; "it" or "function" are not sufficient by themselves as in "the function is the same at $t = 0.3$ and $t = 2.8.$"	$v_P(0.3) = 55 = v_P(2.8)$
In part (a) the response "$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = 0;$ By the Mean Value Theorem, there is a time t between 0.3 and 2.8 when $v_P'(t) = 0$" did not earn the second point because the conditions needed to apply the Mean Value Theorem are not fully verified.	$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = 0;$ v_P is differentiable $\Rightarrow$ v_P is continuous. By the Mean Value Theorem, there is a time t between 0.3 and 2.8 when $v_P'(t) = 0.$
In part (b) not clearly showing a trapezoidal sum as in $\int_0^{2.8} v_P(t)\,dt \approx 8.25 + 18.2 + 14.3 = 40.75$	Showing the trapezoidal sum is necessary; numerical simplification is not required as in $\int_0^{2.8} v_P(t)\,dt \approx$ $\frac{1}{2}\cdot(0 + 55)\cdot 0.3 + \frac{1}{2}\cdot(55 - 29)\cdot 1.4 + \frac{1}{2}\cdot(-29 + 55)\cdot 1.1$
In part (c) the following response uses just two decimal places to report the interval and did not earn the first point; the second point was earned for the definite integral; the third point was not earned due to premature rounding before the integral was evaluated $v_Q(t) \ge 60$ for $1.86 < t < 3.52$	Providing answers that are accurate to three decimal places; retaining full calculator accuracy (storing those values in the calculator and using the stored values) for intermediate calculations $v_Q(t) = 60 \Rightarrow t = A = 1.866181$ or $t = B = 3.519174$ $v_Q(t) \ge 60$ for $A \le t \le B$

Distance traveled $= \int_{1.86}^{3.52} v_Q(t)\,dt = 106.529$	Distance traveled $= \int_{A}^{B} \lvert v_Q(t) \rvert\,dt = 106.109$
In part (d) omitting the differential for an integral can result in an incorrect integrand as in At $t = 2.8$ the position of Q is $\int_0^{2.8} v_Q(t) - 90.$	Including the differential dt to unambiguously declare the integrand At $t = 2.8$ the position of Q is $\int_0^{2.8} v_Q(t)\,dt - 90.$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Particle motion is a standard topic in calculus courses, and many student responses showed this to be relatively familiar territory. Familiarity aside, responses were hampered by challenges in communication and notation. Application of key theorems such as the Mean Value Theorem and Extreme Value Theorem requires that verification of hypotheses be communicated. The precision of mathematical notation carries both power and responsibility. Good notation can succinctly convey a precise statement. Casual notation—indiscriminate use of "=" to connect phrases, or omission of the differential in an integral—can lead to erroneous statements. Teachers can continue to emphasize interpreting and using notation appropriately.

Teachers can also encourage good numerical habits when using a graphing calculator. Intermediate results (for example, solutions to $v_Q(t) = 60$ that are to be used as limits of a definite integral) should be stored in the graphing calculator to retain as much accuracy as possible. Rounding or truncating such values can imperil the accuracy of the final answer. Teachers can ensure that students have sufficient practice with their calculators to have facility with these skills.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- To prepare students to make appropriate use of graphing calculators, it may be useful to review p. 201 of the *AP Calculus AB and BC Course and Exam Description* (*CED*) where you will find a section titled, "Graphing Calculators and Other Technologies in AP Calculus."
- In part (a) students needed to identify and correctly apply the Mean Value Theorem (see Topic 5.1 in the *CED*: Using the Mean Value Theorem). There are several useful resources linked on p. 96 of the *CED*: (1) a classroom resource, "Why We Use Theorem in Calculus"; (2) a discussion from the AP Online Teacher Community on the Mean Value Theorem and Existence Theorems; and (3) an online module found on the professional development tab of the AP Calculus pages at AP Central, "Continuity and Differentiability: Establishing Conditions for Definitions and Theorems." Topic Questions for Topic 5.1 offer scaffolded practice using the Mean Value Theorem.
- In parts (b) and (c) students tended to make communication errors that might have been avoided with careful review and practice of the instructions provided with the free-response questions in the Personal Progress Checks for each unit. In reviewing Unit 1, it might be helpful to read through the instructions with your class, highlighting key points: you must clearly indicate the setup of your question; you must show the mathematical steps necessary to produce your [calculator] results; unless otherwise specified, answers (numeric or algebraic) need not be simplified; if your answer is given as a decimal approximation, it should be correct to three places after the decimal point. Relying on students to carefully read these instructions on AP Exam day is less successful than practice with applying them all year long.

- See Topic 6.2 on p. 116 of the *CED* for a link to a classroom resource, “Reasoning from Tabular Data.”
- In part (d) students who omitted the differential, as in $\int_0^{2.8} v_Q(t) - 90,$ may have needed more practice and feedback on Skill 4.A: Use precise mathematical language. Beginning on p. 214 of the *CED*, you will find “Developing the Mathematical Practices.” On p. 219, you will find sample instructional strategies to develop Skill 4.A. To identify practice items, you might try searching the Question Bank for this skill.

Question AB3/BC3 **Topic:** Graphical Analysis of f / FTC

Max. Points: 9 **Mean Score: AB3:** 2.70; **BC3:** 4.66

What were the responses to this question expected to demonstrate?

In this problem it is given that the function f is continuous on the interval $[-6, 5]$. The portion of the graph of f corresponding to $-2 \le x \le 5$ consists of two line segments and a quarter of a circle, as shown in an accompanying figure. It is noted that the point $(3, 3 - \sqrt{5})$ is on the quarter circle.

In part (a) students were asked to evaluate $\int_{-6}^{-2} f(x)\,dx$, given that $\int_{-6}^{5} f(x)\,dx = 7$. A response should demonstrate the integral property that $\int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx = \int_{-6}^{5} f(x)\,dx$ and use the interpretation of the integral in terms of the area between the graph of f and the x-axis to evaluate $\int_{-2}^{5} f(x)\,dx$ from the given graph.

In part (b) students were asked to evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$. A response should demonstrate the sum and constant multiple properties of definite integrals, together with an application of the Fundamental Theorem of Calculus that gives $\int_{3}^{5} f'(x)\,dx = f(5) - f(3)$.

In part (c) students were asked to find the absolute maximum value for the function g given by $g(x) = \int_{-2}^{x} f(t)\,dt$ on the interval $-2 \le x \le 5$. A response should demonstrate calculus techniques for optimizing a function, starting by applying the Fundamental Theorem of Calculus to obtain $g'(x) = f(x)$, and then using the supplied portion of the graph of f to find critical points for g and to evaluate g at these critical points and the endpoints of the interval.

In part (d) students were asked to evaluate $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$. A response should demonstrate the application of properties of limits, using the supplied portion of the graph of f to evaluate $\lim\limits_{x \to 1} f(x)$ and $\lim\limits_{x \to 1} f'(x)$.

For part (a) see LO FUN-6.A/EK FUN-6.A.2, LO FUN-6.A/EK FUN-6.A.1. For part (b) see LO FUN-6.B/EK FUN-6.B.2. For part (c) see LO FUN-5.A/EK FUN-5.A.2, LO FUN-4.A/EK FUN-4.A.3. For part (d) see LO LIM-1.D/EK LIM-1.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most responses showed knowledge of the requisite integral properties, as well as of the connection of the definite integral to area. However, many of these responses were hampered by poor geometric and algebra/arithmetic skills. Many responses contained errors in calculating $\int_{2}^{5} f(x)\,dx$ as the area of what is left after a quarter circle is removed from a square, or errors in distributing a subtraction across a difference, or arithmetic errors working with fractions.

In part (b) many responses showed understanding that the Fundamental Theorem of Calculus applies to evaluate $\int_{3}^{5} f'(x)\,dx$. Responses that followed a solution route of attempting to find an antiderivative for the entire integrand

$2f'(x) + 4$ often using an incorrect antiderivative. Again, some responses contained errors distributing a subtraction across a difference (in this case, for $f(5) - f(3)$, where $f(3) = 3 - \sqrt{5}$).

In part (c) many responses showed understanding of the Fundamental Theorem of Calculus to obtain that $g'(x) = f(x)$, but some incorrectly stated $g'(x) = f(x) - f(-2)$, or confusingly stated $g'(x) = f(t)\,dt$. Some responses included the correct maximum value for g, but gave only partial justification or had vague or no communication of a valid justification for the maximum.

In part (d) many responses showed understanding that the limit could be evaluated via substitution, but some of these went on to miss the opportunity to earn the point through incorrect attempts to simplify the answer (e.g., misevaluating $\arctan 1$ or arithmetic errors). Some responses showed an attempt to apply the process of L'Hospital's Rule without checking to see that the conditions for L'Hospital's Rule are not satisfied.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) omitting needed parentheses as in $\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$ $= 7 - 2 + 9 - \frac{9\pi}{4}$	$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$ $= 7 - \left(2 + 9 - \frac{9\pi}{4}\right)$
In part (b) incorrect antidifferentiation as in $\int_{3}^{5} (2f'(x) + 4)\,dx = \left[(f(x))^2 + 4x\right]_{x=3}^{x=5}$	Using the Fundamental Theorem of Calculus: $\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$
In part (c) using an argument that only supports a relative maximum, and/or omits consideration of endpoints as in $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$ The absolute maximum occurs at $x = -1$ because $f(x)$ changes from positive to negative there.	$g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$ Checking critical points and endpoints: $g(-2) = 0;\ g(-1) = \frac{1}{2};\ g\left(\frac{1}{2}\right) = -\frac{1}{4};$ $g(5) = 11 - \frac{9\pi}{4}$ Thus the absolute maximum value is $g(5) = 11 - \frac{9\pi}{4}.$
In part (d) making a simplification error as in $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}}$ $= \frac{16}{4 - \pi} = 4 - \frac{16}{\pi}$	$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Students continue to struggle with functions that are presented in ways other than analytically, such as the general function f which is specified by a graph in this problem. Teachers can find ways to provide as much experience with this topic as practical, such as using released free-response questions from previous AP Calculus Exams. Students also need guidance and practice in organizing work and calculations into a coherent narrative for a problem's solution. In part (a), for example, scattered computations posed a challenge (in some cases, an insurmountable challenge) to award partial credit. Also, the exam instructions indicate that justifications require *mathematical reasons*. For example, it is not sufficient in part (c) to support the answer that $g(5) = 11 - \frac{9\pi}{4}$ is the absolute maximum value of g by stating that "the most area is at the right endpoint." Teachers are urged to find opportunities to guide students toward constructing valid justifications as a reoccurring thread throughout the course. Finally, too many students had responses that did not earn a particular point because of algebra or arithmetic errors "simplifying" answers where such simplification is not required. It is important for students to hone their algebra and arithmetic skills. These skills are necessary to match options for questions in the multiple-choice section of the AP Calculus Exam, but students can be coached to refrain from simplifying answers on the free-response section to avoid missing opportunities to earn points that would have been earned otherwise.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- Students need practice with calculus content and skills and with the geometry and arithmetic necessary to determine a correct answer in part (a). Topic Questions from Topic 6.6 of the *AP Calculus AB and BC Course and Exam Description* (*CED*) offer the practice students need on their way to mastery. Distractor analysis or error analysis (see p. 206 of the *CED*) might also be useful strategies for students who need to improve their precalculus skills.
- While students often move quickly from introduction to the Fundamental Theorem of Calculus to finding antiderivatives or evaluating definite integrals, it is important to revisit the theorem itself so that students can develop the deeper understanding they need to avoid the common errors seen in part (b) of this question. The Question Bank is a good resource for items you can use on quizzes or tests, whether in Unit 6 or as review items in later units. If you want to find items featuring the Fundamental Theorem of Calculus, try searching by Topics 6.4–6.7.
- As in Question AB1/BC1 part (c), some students relied on local arguments rather than justifying an absolute maximum. On p. 97 of the *CED*, see links to several useful resources. The classroom resource "Extrema" might be especially relevant.

Question AB4/BC4 **Topic:** Modeling with Separable Differential Equation

Max. Points: 9 **Mean Score: AB4:** 1.81; **BC4:** 3.56

What were the responses to this question expected to demonstrate?

The context for this problem is a cylindrical barrel with a diameter of 2 feet that contains collected rainwater, some of which drains out through a valve in the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h},$ where h is measured in feet, and t is measured in seconds.

In part (a) students were asked to find the rate of change of the volume of water in the barrel with respect to time when $h = 4$ feet. A response should use the geometric relationship between the volume V of water in the barrel and height h and incorporate the given expression for $\frac{dh}{dt}$.

In part (b) students were asked to determine whether the rate of change of the height of water in the barrel is increasing or decreasing when $h = 3$ feet. A response should demonstrate facility with the chain rule to differentiate $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ with respect to time to obtain $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}.$ Because $\frac{d^2h}{dt^2} > 0,$ a response should conclude that the rate of change of the height of the water in the barrel is increasing.

In part (c) students were given that the height of the water is 5 feet at time $t = 0$ and then asked to use the technique of separation of variables to find an expression for h in terms of $t.$ A response should demonstrate the application of separation of variables to solve the differential equation $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ for h and then incorporate the initial condition that $h(0) = 5$ to find the particular solution $h(t)$ to the differential equation.

For part (a) see LO CHA-3.D/EK CHA-3.D.1, LO CHA-3.E/EK CHA-3.E.1. For part (b) see LO FUN-4.E/EK FUN-4.E.2. For part (c) see LO FUN-7.D/EK FUN-7.D.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-7.E/EK FUN-7.E.1. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) many responses showed recognition of this as a related rates problem but faltered finding the derivative of $\pi r^2 h$ with respect to $t.$ Dealing with the radius r was problematic. Some responses exhibited confusion between diameter and radius, using $r = 2;$ others included treatment of r as a potentially non-constant variable but omitted $\frac{dr}{dt}$ in differentiating $V = \pi r^2 h$ with respect to $t;$ and still others imposed a geometrical relationship that yielded an expression for r in terms of $h.$

In part (b) only a minority of responses showed understanding that "with respect to time" in the problem called for a derivative with respect to $t.$ Many responses based an answer upon substituting $h = 3$ into the given expression for $\frac{dh}{dt}$ or into the derivative of that expression with respect to $h,$ $\frac{d}{dh}\left(\frac{dh}{dt}\right).$

In part (c) many responses showed skills to separate variables correctly and included the constant of integration at the appropriate step. Some responses showed difficulties in dealing with a differential equation in which the independent variable t does not explicitly appear. Also, some responses contained errors in an antiderivative for $\frac{1}{\sqrt{h}}$, and some included algebraic mistakes when solving for h in terms of t.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) treating r as a variable but giving an incomplete derivative of the volume of water with respect to time as in $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi rh + \pi r^2 \frac{dh}{dt}$	Because $r = 1$, $V = \pi \cdot 1^2 \cdot h = \pi h$. Thus $\frac{dV}{dt} = \pi \frac{dh}{dt}$.
In part (b) ambiguous derivative notation can result in an incomplete derivative of h with respect to time as in $h'' = -\frac{1}{20\sqrt{h}}$	$\frac{d^2h}{dt^2} = \frac{d}{dt}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$
In part (c) an incorrect antiderivative for $\frac{1}{\sqrt{h}}$ as in $\int \frac{1}{\sqrt{h}}\, dh = \int \left(-\frac{1}{10}\right) dt \Rightarrow \ln\sqrt{h} = -\frac{1}{10}t + C$	$\int \frac{1}{\sqrt{h}}\, dh = \int \left(-\frac{1}{10}\right) dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$
In part (c) introducing the constant of integration as an afterthought as in $\int \frac{1}{\sqrt{h}}\, dh = \int \left(-\frac{1}{10}\right) dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t$ $\Rightarrow 4h = \frac{1}{100}t^2 + C$ $4 \cdot 5 = 0 + C \Rightarrow C = 20$ $h = \frac{1}{4}\left(\frac{1}{100}t^2 + 20\right)$	Introducing the constant of integration in the step where antiderivatives appear: $\int \frac{1}{\sqrt{h}}\, dh = \int \left(-\frac{1}{10}\right) dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$ $2\sqrt{5} = 0 + C \Rightarrow C = 2\sqrt{5}$ $h = \frac{1}{4}\left(-\frac{1}{10}t + 2\sqrt{5}\right)^2$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

The AP Calculus exam often provides a new opportunity for AP students to interpret, apply, and/or solve a differential equation in context. These are rich problems. Questions that derive information directly from the differential equation, such as parts (a) and (b) in this question, show how information can be learned about a quantity without an explicit formula for that quantity (and even if we lack the tools to find an explicit formula). From the context, students deal with mnemonic variables other than just the generic x, y, or t. Solving a separable differential equation draws upon algebraic manipulation and antidifferentiation skills, and in understanding and applying an initial condition. All of this is facilitated by good notational understanding and habits. Teachers can target each of these skills at various points throughout the course and find multiple opportunities to use released free-response questions from previous AP Calculus Exams, such as this one as a capstone and to drive home that these individual skills provide an important foundation for solving more complex problems.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- The Topic Questions for Topics 4.1, 4.4, and 4.5 nicely scaffold student understanding of related rates and mastery of corresponding skills required in part (a). In Topic 4.1, the Topic Questions give feedback on understanding of the relationship between rates and derivatives; in Topic 4.4, the questions see how students are doing at identifying the need for the chain rule and applying it in related rates problems. Questions in Topic 4.5 pull it all together. Topic Questions were designed to address common errors on the AP Calculus Exams that have been identified by previous AP Chief Readers and should help to prepare students with the understanding and mastery to avoid similar errors in the future.
- Communication and Notation are linked not only in part (b) of this question but also in all elements of the course. "Notation Read Aloud" is an instructional strategy to help students to master the connections between notation and meaning and is found on p. 209 of the *AP Calculus AB and BC Course and Exam Description* (*CED*).
- Students who had difficulty antidifferentiating a radical expression in this question may have struggled to identify the rational exponent that would have helped them to choose the power rule. For practice items working with radical expressions, try searching the Question Bank by "type of expression."
- To prepare for solving separable differential equations, consider using the classroom resource linked on p. 140 of the *CED*: "Differential Equations."

Question AB5 **Topic:** Area/Volume

Max. Points: 9 **Mean Score:** 3.21

What were the responses to this question expected to demonstrate?

In this problem R is identified as the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$.

In part (a) students were asked to find the area of R. A response should demonstrate the area interpretation of definite integrals and compute the area of R as $\int_0^2 (h(x) - g(x))\,dx$. Students should find an antiderivative for $h(x) - g(x)$ and apply the Fundamental Theorem of Calculus to evaluate the integral.

In part (b) students were asked to find the volume of a solid having R as its base and for which at each x, the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. As before, the Fundamental Theorem of Calculus should be employed to evaluate $\int_0^2 A(x)\,dx$.

In part (c) students were asked to write an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. In this case, however, the cross section at x is a “washer” with outer radius $6 - g(x)$ and inner radius $6 - h(x)$, so the area of the cross section at x can be expressed using the familiar formula for the area of a circle.

For part (a) see LO CHA-5.A/EK CHA-5.A.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (b) see LO CHA-5.B/EK CHA-5.B.3, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO CHA-5.C/EK CHA-5.C.4. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most responses showed understanding of the connection between a definite integral and area. However, many responses contained errors in the skills needed to execute a solution to the problem. The principal calculus deficiency was dealing with $\frac{\pi}{2}$ in $3\cos\left(\frac{\pi}{2}x\right)$ when finding an antiderivative. In addition, responses showed errors in discerning which of g or h corresponds to the upper boundary, and which to the lower boundary, of region R, and a multiple of algebra and notational errors, such as omitting needed parentheses, or failure to distribute a subtraction across a sum or difference. Some responses included an expansion of $(x-1)^2$ before antidifferentiating, introducing opportunities for algebraic pitfalls.

In part (b) many responses showed understanding of the need to integrate a cross-sectional area, but attempted to construct the area of a cross section, not realizing that it had been supplied by the given function A. Thus many responses included a construction of a function for the area of a cross section, often using $A(x)$ as one dimension of the cross section.

In part (c) many responses were correct. Some responses did not contain the needed factor of π or included it within the integrand as a factor of only one of the two terms. Some responses used an axis of rotation other than $y = 6$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) incorrectly antidifferentiating $3\cos\left(\frac{\pi}{2}x\right)$ as in $\pi\int_0^2 (h(x) - g(x))\,dx$ $= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + 3\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$	$\pi\int_0^2 (h(x) - g(x))\,dx$ $= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$
In part (b) incomplete transformation of a definite integral under substitution as in Let $u = x + 3$. $\int_0^2 \frac{1}{x+3}\,dx = \int_0^2 \frac{1}{u}\,du = \ln u\Big\vert_{u=0}^{u=2} = \ln 2 - \ln 0$	Let $u = x + 3$. $\int_0^2 \frac{1}{x+3}\,dx = \int_3^5 \frac{1}{u}\,du = \ln u\Big\vert_{u=3}^{u=5} = \ln 5 - \ln 3$
In part (c) not rotating about the line $y = 6$ as in $\pi\int_0^2 \left((h(x))^2 - (g(x))^2\right)dx$	$\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right)dx$
In part (c) mistaking the inner radius for the outer radius: $\pi\int_0^2 \left((6 - h(x))^2 - (6 - g(x))^2\right)dx$	$\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right)dx$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

A slight variation from a common presentation of a problem type sometimes poses a problem for students who memorize procedures without understanding underlying principles. An example is part (b) in this problem where the cross-sectional area $A(x) = \frac{1}{x+3}$ is given, as opposed to constructed as the area of a square, triangle, or other figure. This issue arose in part (c), as well, where students often viewed the "washer method" as a distinct problem type unrelated to finding volume from known cross sections. In the latter case, teachers can emphasize using a figure to describe the cross section, from which it should be clear what the radii are and which is larger. Encouraging students to explain how the parts of an integral give a volume, either to a peer or in writing, can help reinforce the concepts underlying a volume-finding procedure.

Another issue is the technique of substitution in a definite integral. This occurred in many responses in part (a) for $u = \frac{\pi}{2}x$ and in part (b) for $u = x + 3.$ Many responses included a transformation of the integrand in terms of the substituted variable u but failed to adjust the limits of the integral accordingly. Unless explicitly noted otherwise, the limits on a definite integral are assumed to correspond to the variable of integration. Teachers need to emphasize this; an integral expression with integrand in terms of u and un-labeled limits corresponding to x is incorrect and often results in a point not being earned.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- On page 123 of the *AP Calculus AB and BC Course and Exam Description* (*CED*), see the link to a resource to better prepare students for using substitution as an integration technique: "Applying Procedures for Integration by Substitution."
- See the link to a classroom resource to prepare students to find volumes of solids of revolution on p. 160 of the *CED*: "Volumes of Solids of Revolution."
- The Topic Questions for Topics 8.7–8.12 scaffold student practice and give teachers and students feedback on which content or skills might need particular attention.

Question AB6 **Topic:** Analysis of Functions with L'Hospital and Squeeze Theorem

Max. Points: 9 **Mean Score:** 2.84

What were the responses to this question expected to demonstrate?

This problem introduces three twice-differentiable functions f, g, and h. It is given that $g(2) = h(2) = 4$, and the line $y = 4 + \frac{2}{3}(x - 2)$ is tangent at $x = 2$ to both the graph of g and the graph of h.

In part (a) students were asked to find $h'(2)$. A response should demonstrate the interpretation of the derivative as the slope of a tangent line and answer with the slope of the line $y = 4 + \frac{2}{3}(x - 2)$.

In part (b) the function a given by $a(x) = 3x^3h(x)$ is defined, and students were asked for an expression for $a'(x)$ and the value of $a'(2)$. A response should demonstrate facility with the product rule for differentiation.

In part (c) it is given that the function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$ and that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Students were then asked to find $f(2)$ and $f'(2)$. A response should observe that the differentiability of h implies that h is continuous so that $\lim_{x \to 2} h(x) = h(2) = 4$. Because $\lim_{x \to 2} \left(x^2 - 4\right) = 0$, and $\lim_{x \to 2} h(x)$ can be evaluated, as is given, it must be that $\lim_{x \to 2} \left(1 - (f(x))^3\right) = 0$, as well. Using properties of limits, students could conclude that $\lim_{x \to 2} f(x) = 1$. Finally, an application of L'Hospital's Rule to $\lim_{x \to 2} \dfrac{x^2 - 4}{1 - (f(x))^3}$, combined with the chain rule to differentiate $(f(x))^3$, yields an equation that can be solved for $f'(2)$.

In part (d) students were given that $g(x) \le h(x)$ for $1 < x < 3$ and that k is a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Students were asked to decide, with justification, whether k is continuous at $x = 2$. A response should observe that the differentiability of g and h implies that these functions are continuous, so the limits as x approaches 2 of each of g and h match the value $g(2) = h(2) = 4$. From the inequality $g(2) \le k(2) \le h(2)$ it follows that $k(2) = 4$, and the squeeze theorem applies to show that k is continuous at $x = 2$.

For part (a) see LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-3.B/EK FUN-3.B.1. For part (c) see LO LIM-2.A/EK LIM-2.A.2, LO LIM-4.A/EK LIM-4.A.2. For part (d) see LO LIM-1.E/EK LIM.1.E.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 3: Justification, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most responses correctly found $h'(2)$ to be the slope of the given tangent line to the graph of h at $x = 2$.

In part (b) most responses showed understanding that a product rule is needed to find $a'(x)$, although some of these responses contained errors in one or more parts of the product rule expression.

In part (c) most responses revealed an association between using L'Hospital's Rule and a ratio of derivatives, but many responses contained errors in execution and/or presentation of the L'Hospital's Rule process. Many responses included a statement that could be interpreted as $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4,$ but few responses contained an explicit attribution of this to the given information that h is differentiable and continuous, and $h(2) = 4.$ Some responses revealed the mechanics of L'Hospital's Rule without exploring the conditions under which it applies. These responses concluded that $4 = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)} = \frac{4}{-3(f(2))^2 \cdot f'(2)}$ and so $(f(2))^2 \cdot f'(2) = -\frac{1}{3},$ but overlooked that this process required that $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$ is an indeterminate form so were not able to proceed further to find $f(2)$ and then $f'(2).$

In part (d) many responses showed partial understanding of continuity and concluded that k is continuous at $x = 2.$ However, most of these responses lacked a complete justification for the conclusion, either failing to mention that g and h are continuous or omitting a discussion of limits.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) some responses presented $4,$ the value of $h(2),$ as the value of $h'(2).$	$h'(2) = \frac{2}{3},$ the slope of the line tangent to the graph of h at $x = 2.$
In part (b) dropping the exponent from the derivative of $3x^3$ as in $a'(x) = 9xh(x) + 3x^3h'(x)$	$a'(x) = 9x^2h(x) + 3x^3h'(x)$
In part (c) not recognizing the indeterminate form as in $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \frac{0}{1 - (f(2))^3} = 4$ $\Rightarrow 0 = 4 \cdot \left(1 - (f(2))^3\right)$ $\Rightarrow f(2) = 1$	f is differentiable $\Rightarrow$ f is continuous. In particular $\lim_{x \to 2} f(x) = f(2).$ $\lim_{x \to 2} \left(x^2 - 4\right) = 0$ and $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$ can be evaluated using L'Hospital's Rule so $\lim_{x \to 2} \left(1 - (f(x))^3\right) = \left(1 - (f(2))^3\right) = 0.$ Thus $f(2) = 1.$
In part (c) using the symbol "$\frac{0}{0}$" in the context of a numerical value as in $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \frac{0}{0}$	$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$ is a $\frac{0}{0}$ indeterminate form
In part (d) concluding continuity without any reference to limits as in	Because g and h are differentiable, g and h are continuous, so $\lim_{x \to 2} g(x) = g(2) = 4$ and $\lim_{x \to 2} h(x) = h(2) = 4.$

k is continuous at $x = 2$ by the squeeze theorem because $g(x) \le k(x) \le h(x)$ and $g(2) = h(2) = 4$, so $k(2) = 4$.	Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x\to 2} k(x) = 4$. Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$. Because $\lim_{x\to 2} k(x) = 4 = k(2)$, k is continuous at $x = 2$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

In terms of specific skills, some responses showed a lack of comfort with derivative rules applied to expressions that are a mixture of analytical and general functions, such as the product rule applied to find $a'(x)$ in part (b) or the chain rule for $\frac{d}{dx}\left(1 - (f(x))^3\right)$ in part (c). Teachers can remember to include general functions when creating derivative practice opportunities for their students.

The biggest issue highlighted for teachers by responses to this problem is the overall performance on the topic of limits. This spanned from poor or lacking limit notation to the fundamental role limits play for continuity. In part (c) students were asked to evaluate a limit, but many responses showed a lack of understanding of where the operator "$\lim_{x\to 2}$" should appear, either dispensing with it altogether, or including it beyond the step at which the limit had been taken. In part (d) a high portion of the responses seemed not to associate continuity with limits. In this part, the problem statement includes the admonition to "justify your answer." General instructions for the free-response section of the exam include the sentence, "[j]ustifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied." In this case the relevant theorem is the squeeze theorem, and both its conditions and its conclusion are expressed in terms of limits. To apply the squeeze theorem here, we need to know that $\lim_{x\to 2} g(x)$ and $\lim_{x\to 2} h(x)$ are the same. To evaluate these limits, we need to know that g and h are continuous, so that these limits equal the given values of the respective functions at $x = 2$. The conclusion of the squeeze theorem gives the value for a limit of the "between" function (here played by k). The reason this limit is relevant is because the problem asks about the continuity of k at $x = 2$: does k have a limit as x approaches 2, does k have a value at $x = 2$, and do these two agree? Limits permeate part (d) of this problem, as they do calculus. You can't define the derivative or a definite integral without limits. Students need to understand and work with limits beyond evaluating limits via algebraic manipulations. This likely requires guided discussion opportunities to refine student notions of limits (e.g., perhaps starting with why, in evaluating $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$, it is valid to replace this by $\lim_{x\to 1}(x + 1)$). Part (d) is a an excellent vehicle for class discussion, both emphasizing for students what is required for "justifications," exploring limits and continuity, and seeing what is needed to evaluate a limit in a setting with general functions (such as $\lim_{x\to 2} g(x)$, $\lim_{x\to 2} h(x)$, and $\lim_{x\to 2} k(x)$ in this problem).

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- In part (c) the "Preparing for the AP Exam" section of the Unit Guide on p. 81 of the *AP Calculus AB and BC Course and Exam Description* (*CED*) is a useful resource for preparing students to present their work on

questions involving L'Hospital's Rule. On p. 90 of the *CED*, also see links to several resources relating to teaching and learning to understand, apply, and present work involving L'Hospital's Rule.

- On p. 45 of the *CED*, see a link to a resource on the AP Online Teacher Community: "Video on Continuity." This might help to prepare students who lack understanding of how to argue continuity using the definition. Also, on p. 34 of the *CED*, see "Discussion Groups," a sample instructional strategy that might help to prepare students. As always, ongoing practice is important. Students might struggle with this concept because reasoning with definitions and theorems can take some time to fully master and because the definition of continuity was explained during the first part of the school year.

Question BC2 **Topic:** Polar Area

Max. Points: 9 **Mean Score:** 2.60

What were the responses to this question expected to demonstrate?

In this problem a region S is shown in an accompanying figure, and S is identified as the region enclosed by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}.$

In part (a) students were asked to find the area of $S.$ A response should demonstrate knowledge of the form of the integral that gives the area of a simple polar region and evaluate $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta$ using the numerical integration capability of a graphing calculator.

In part (b) students were asked for the average distance from the origin to a point on the polar curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}.$ A response should observe that the distance from the origin to a point on the polar curve is given simply by $r(\theta)$ and then should demonstrate that the average value of $r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is given by dividing the definite integral of $r(\theta)$ across the interval by the width of the interval. The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

In part (c) m denotes the positive slope of a line through the origin that divides the region S into two regions of equal areas. Students were asked to write an equation involving one or more integrals whose solution gives the value of $m.$ A response should express the polar angle θ formed by the line and the polar axis in terms of m (namely, $\theta = \tan^{-1} m$) and use this as an upper limit in an integral that corresponds to polar area within an equation satisfying the given requirements.

In part (d) it is given that $A(k)$ represents the area of the portion of region S that is inside the circle $r = k\cos\theta,$ and students were asked for the value of $\lim_{k\to\infty} A(k).$ A response should observe that any point to the right of the y-axis will eventually be inside the circle $r = k\cos\theta$ for k sufficiently large. Thus $\lim_{k\to\infty} A(k)$ is the area of the portion of S inside the first quadrant, computed as $\frac{1}{2}\int_0^{\frac{\pi}{2}} (r(\theta))^2 \, d\theta.$ The resulting integral expression should be evaluated using the numerical integration capability of a graphing calculator.

For part (a) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1, LO LIM-5.A/EK LIM-5.A.3. For part (c) see LO CHA-5.D/EK CHA-5.D.1. For part (d) see LO CHA-5.D/EK CHA-5.D.2, LO LIM-5.A/EK LIM-5.A.3. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) many responses contained a correct integral setup for the area of the polar region S and included a correct numerical value for the integral as obtained from a graphing calculator. Some responses had an integral setup that would match an area in rectangular $r\theta$-coordinates.

In part (b) many responses showed understanding of an average value integral setup, but many of these responses showed difficulty with the integrand measuring distance to the origin in a polar environment. Some responses attempted to derive this using the distance formula from rectangular coordinates and the conversion $x = r\cos\theta,$ $y = r\sin\theta,$ with some of these attempts being successful.

In part (c) some responses contained an equation equating polar areas, but very few responses showed an understanding of the interface between polar coordinates and the rectangular coordinates to connect the slope m of a line through the origin to the polar angle θ that the line forms with the polar axis.

In part (d) some responses showed understanding of the growth of the circles $r = k\cos\theta$ as k increases, some depicting this via graphical sketches and concluding that $\lim_{k\to\infty} A(k)$ represents the area of the portion of S to the right of the y-axis. Responses that included an analytical attempt to specify the angle of intersection of $r = k\cos\theta$ and $r = 3\sqrt{\theta}\sin\left(\theta^2\right)$ did not find this to be a fruitful solution route.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) using a rectangular setup for polar area as in $\text{Area} = \int_0^{\sqrt{\pi}} r(\theta)\,d\theta$	$\text{Area} = \frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\,d\theta$
In part (b) some responses interpreted distance as length of a curve and applied a formula from rectangular coordinates for the integral as in $\text{Average Distance} = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} \sqrt{1+(r'(\theta))^2}\,d\theta = 4.089$	The distance from the origin to a point on the polar curve $r = r(\theta)$ is $r(\theta)$. $\text{Average Distance} = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} r(\theta)\,d\theta = 1.580$
In part (c) using the slope m as a limit of integration where the corresponding angle is expected as in $\int_0^m \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta = \int_m^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta$	For a line through the origin with slope m forming an angle θ with the polar axis, $m = \tan\theta$ so $\theta = \tan^{-1} m$. $\int_0^{\tan^{-1} m} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta = \int_{\tan^{-1} m}^{\sqrt{\pi}} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta$
In part (d) treating the region whose area is measured by $A(k)$ to be a region between curves as in $\lim_{k\to\infty} A(k) = \lim_{k\to\infty}\int_0^{\sqrt{\pi}} \left((k\cos\theta)^2 - \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\right) d\theta$	As $k \to \infty$ the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis. $\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 d\theta$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

The polar (versus rectangular) perspective can be a difficult one for students to acquire. Some questions stay in a purely polar realm, such as finding the area of S in part (a), while others bridge the divide between polar and rectangular, such as relating slope of a line through the origin to a polar angle in part (c). Any activity, individual or group, which increases student comfort and competence with polar coordinates will pay dividends later.

One notational aspect that impacted scores on this problem was dealing carefully with parentheses. An integrand like $(r(\theta))^2 = \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2$ has several pairs of parentheses to manage, and a couple instances of squaring that sometimes wandered in and out of parentheses, and sometimes a left parenthesis would open and not be eventually paired with a closing right parenthesis. It is understandable that haste on a timed, high stakes exam could cause some omissions, though teachers might find a time for a team activity comparing various parenthesizations of expressions, or incompletely parenthesized expressions, to emphasize the importance of parentheses in a carefully notated expression.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- To increase student comfort and competence with polar coordinates, consider using Sample Activity 3 found on p. 167 of the *AP Calculus AB and BC Course and Exam Description* (*CED*): "Create Representations."
- Continue to practice questions requiring notational fluency, along with questions involving the product rule and the chain rule, which may be needed to differentiate $x = r\cos\theta$ and $y = r\sin\theta$ with respect to θ.
- Similarly, continuing to review differentiation for parametric equations will help to develop the mastery students need to transfer understanding to polar parameterizations. Topic Questions and the Question Bank are good resources for practice items. For additional practice with differentiating and integrating parametric equations, consider using the classroom resource linked on p. 169 of the *CED*: "Vectors." This module is an excellent resource for teaching and reviewing Topics 9.1–9.6.
- To build understanding of how to use definite integrals to find areas of regions defined by polar functions, students need to understand the connection between setting up a sum of areas of rectangles and setting up a sum of areas of sectors. Consider adapting Sample Activity 2 on p. 148 of the *CED* ("Round Table") to include setting up integrals to find polar areas. Revisiting this activity both builds new understanding and reinforces past learning.

Question BC5 **Topic:** Analysis of Functions/Partial Fractions/Improper Integral

Max. Points: 9 **Mean Score:** 4.05

What were the responses to this question expected to demonstrate?

This problem deals with a family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

In part (a) students were asked to find the positive value of k such that the slope of the line tangent to the graph of f at $x = 0$ equals 6. A response should demonstrate differentiation rules to find $f'(x)$ and then identify $f'(0)$ as the slope of the line tangent to the graph of f at $x = 0$, so the k can be found by solving $f'(0) = 6$.

In part (b) students were asked to evaluate $\int_0^1 f(x)\,dx$ in the case where $k = -8$. A response should demonstrate that, with $k = -8$, $f(x)$ can be expressed using partial fractions as $f(x) = \dfrac{1}{x^2 - 2x - 8} = \dfrac{\frac{1}{6}}{x - 4} - \dfrac{\frac{1}{6}}{x + 2}$. Then $\int_0^1 f(x)\,dx$ can be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (c) students were asked to evaluate $\int_0^2 f(x)\,dx$ or show that it diverges, in the case where $k = 1$. A response should note that $x^2 - 2x + 1 = (x - 1)^2$, so the graph of f has a vertical asymptote at $x = 1$ and $\int_0^2 f(x)\,dx$ is an improper integral. Thus $\int_0^2 f(x)\,dx$ is the sum $\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$, providing each of the summands converges. Expressing either summand as a one-sided limit of a proper integral, a response should demonstrate that the summand diverges and conclude that $\int_0^2 f(x)\,dx$ diverges.

For part (a) see LO FUN-3.C/EK FUN-3.C.1, LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-6.F.b/EK FUN-6.F.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO LIM-6.A/EK LIM-6.A.1, LO FUN-6.C/EK FUN-6.C.2, LO LIM-6.A/EK LIM-6.A.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) responses generally showed understanding of the slope interpretation of the derivative of a function. Issues that arose were either differentiation errors or algebra errors in solving $f'(0) = 6$ for k.

In part (b) many responses showed understanding and execution of a partial fraction decomposition for the integrand $\dfrac{1}{x^2 - 2x - 8}$. The most common antidifferentiation error was to use $\ln(x - 4)$ to evaluate $\int_0^1 \dfrac{1}{x - 4}\,dx$ instead of $\ln|x - 4|$ or $\ln(4 - x)$.

In part (c) most responses did not account for the singularity in the integrand at $x = 1$. Of those that did, many responses expressed the integrals $\int_0^1 \dfrac{1}{(x-1)^2}\,dx$ and $\int_1^2 \dfrac{1}{(x-1)^2}\,dx$ as limits of proper integrals, but few expressed these as one-sided limits.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (a) some responses complicated the derivative by using the quotient rule, and then had a parentheses error as in $f'(x) = \dfrac{x^2 - 2x + k(0) - (2x-2)}{\left(x^2 - 2x + k\right)^2}$	$f(x) = -\left(x^2 - 2x + k\right)^{-1}$ $f'(x) = -\left(x^2 - 2x + k\right)^{-2} \cdot (2x - 2)$ $= \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$
In part (b) using an antiderivative for $\dfrac{1}{x-4}$ that is not defined on $[0, 1]$ as in $\int_0^1 f(x)\,dx = \dfrac{1}{6}\int_0^1 \left(\dfrac{1}{x-4} - \dfrac{1}{x+2}\right) dx$ $= \dfrac{1}{6} \cdot \left[\ln(x-4) - \ln(x+2)\right]_{x=0}^{x=1}$	$\int_0^1 f(x)\,dx = \dfrac{1}{6}\int_0^1 \left(\dfrac{1}{x-4} - \dfrac{1}{x+2}\right) dx$ $= \dfrac{1}{6} \cdot \left[\ln\lvert x-4\rvert - \ln\lvert x+2\rvert\right]_{x=0}^{x=1}$
In part (c) not recognizing the indeterminate form as in $\int_0^2 \dfrac{1}{(x-1)^2}\,dx = \left[-\dfrac{1}{x-1}\right]_{x=0}^{x=2} = -1 - 1 = -2$	$\int_0^2 \dfrac{1}{(x-1)^2}\,dx$ $= \lim\limits_{b\to 1^-} \int_0^b \dfrac{1}{(x-1)^2}\,dx + \lim\limits_{a\to 1^+} \int_a^2 \dfrac{1}{(x-1)^2}\,dx$ $= \lim\limits_{b\to 1^-} \left(-\dfrac{1}{x-1}\Big\vert_{x=0}^{x=b}\right) + \lim\limits_{a\to 1^+} \left(-\dfrac{1}{x-1}\Big\vert_{x=a}^{x=2}\right)$ $= \lim\limits_{b\to 1^-} \left(-\dfrac{1}{b-1} - 1\right) + \lim\limits_{a\to 1^+} \left(-1 + \dfrac{1}{a-1}\right)$ The integral diverges because $\lim\limits_{b\to 1^-} \left(-\dfrac{1}{b-1} - 1\right) = +\infty.$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Many of the errors in parts (a) and (b) were algebra errors, false logarithm “rules,” or errors in presentation. While keeping the focus of the course on calculus, teachers should encourage good algebraic habits (and discourage poor habits) consistently through the year.

The principal calculus deficiency exhibited by responses to this problem was not recognizing an improper integral. Teachers can encourage checking that a definite integral is proper (or not) up front as part of the routine of considering a new integral expression. Another habit to encourage is that when an integral is rewritten as a sum of improper integrals for which the singularity occurs at one of the limits of integration, the corresponding limit expressions are one-sided limits.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- Forgetting to consider whether an interval of integration includes a vertical asymptote, for example, is a common student error. It is worth investing the time in making sure that students understand the implications of various types of discontinuities on selection of integration technique. A strategy like "Odd One Out" on p. 209 of the *AP Calculus AB and BC Course and Exam Description* (*CED*) might be adapted to help students to recognize improper integrals.
- Topic 6.13, Evaluating Improper Integrals, offers an opportunity to review notation and evaluation of one-sided limits. Mastery of this topic will also lay the foundation for an understanding of divergence of series and the Integral Test for Convergence, Topic 10.4.

Question BC6 **Topic:** Taylor Polynomials with Alternating Series Error Bound

Max. Points: 9 **Mean Score:** 3.75

What were the responses to this question expected to demonstrate?

In this problem a function f is presented that has derivatives of all orders for all real numbers x. A figure showing a portion of the graph of f and a line tangent to the graph of f at $x = 0$ is given, as is a table showing values for $f^{(2)}(0)$, $f^{(3)}(0)$, and $f^{(4)}(0)$.

In part (a) students were asked to write the third-degree Taylor polynomial for f about $x = 0$. A response should demonstrate that terms of the Taylor polynomial have the form $\frac{f^{(n)}(0)}{n!}x^n$, determine $f(0)$ and $f'(0)$ from the given graph, and find values for $f''(0) = f^{(2)}(0)$ and $f^{(3)}(0)$ in the table to construct the requested Taylor polynomial.

In part (b) students were asked to write the first three nonzero terms of the Maclaurin series for e^x and to provide the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$. A response should state that the Maclaurin series for e^x starts with the terms $1 + x + \frac{1}{2!}x^2 + \cdots$ and then form the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ using the terms of degree at most 2 in the product $\left(1 + x + \frac{1}{2!}x^2\right) \cdot T_3(x)$, where $T_3(x)$ is the Taylor polynomial found in part (a).

In part (c) students were asked to use the Taylor polynomial found in part (a) to approximate $h(1)$, where $h(x) = \int_0^x f(t)\,dt$. A response should demonstrate that $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 T_3(t)\,dt$ where $T_3(x)$ is the Taylor polynomial found in part (a). $\int_0^1 T_3(t)\,dt$ should be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (d) it is given that the Maclaurin series for h converges to $h(x)$ everywhere and that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Students were asked to use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45. A response should demonstrate that the error in the approximation is bounded by the magnitude of the first omitted term of the series for $h(1)$. This term is found by integrating the fourth-degree term of the Taylor series for f about $x = 0$ across the interval $[0, 1]$. Computing this term demonstrates the desired error bound.

For part (a) see LO LIM-8.A/EK LIM-8.A.1. For part (b) see LO LIM-8.F/EK LIM-8.F.2, LO LIM-8.G/EK LIM.8.G.1. For part (c) see LO LIM-8.G/EK LIM-8.G.1. For part (d) see LO LIM-8.C/EK LIM-8.C.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) many responses showed understanding of the form of a Taylor polynomial. Most of these responses resourced $f(0)$ and $f'(0)$ from the graph and $f''(0)$ and $f^{(3)}(0)$ from the table, although getting information from the graph tended to be more of a challenge. Some responses presented a Taylor series for a Taylor polynomial, and some other responses exhibited confusion about "third-degree," presenting either just three nonzero terms or only the third-degree term.

In part (b) many responses showed knowledge of the first three terms of the Maclaurin series for e^x. Attempts to assemble the second-degree Taylor polynomial for $e^x f(x)$ followed one of two paths: multiplying the series for e^x and the series for $f(x)$ or direct computation of the coefficients from derivatives of $e^x f(x)$. Some responses on the former path suffered from errors in multiplying polynomials or from not collecting terms to give a Taylor polynomial. The latter path required repeated product rules with an expression involving the general function f.

In part (c) many responses showed understanding of integrating the polynomial from part (a) over the interval $[0, 1]$ to give an approximation for $h(1)$. Some responses, however, showed an interpretation of the instructions to merely evaluate the polynomial from part (a) for the approximation. These seemed to miss the concept that a function's Taylor polynomials are approximations to the function in the vicinity of where the Taylor polynomial is centered.

In part (d) many responses showed understanding that the alternating series error bound involves the first omitted term of the series for $h(1)$. However, some responses attempted to use the Lagrange error bound, contrary to the given instruction. Many responses showed a weak understanding of "error bound," either stating the error equals 0.45 or producing a value without identifying it as a bound on the error of the approximation.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
In part (b) after finding $P(x) = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$ in part (a) and $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, computing the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ as $(3 \cdot 1) + (-2 \cdot 1)x + \left(\frac{3}{2} \cdot \frac{1}{2}\right)x^2 = 3 - 2x + \frac{3}{4}x^2$	The second-degree Taylor polynomial for $e^x f(x) = \left(1 + x + \frac{1}{2}x^2 + \cdots\right)\left(3 - 2x + \frac{3}{2}x^2 + \cdots\right)$ about $x = 0$ is $3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 = 3 + x + x^2$
In part (c) using the Taylor polynomial from part (a) without integrating as in $h(1) \approx 3 - 2 + \frac{3}{2} - \frac{23}{12}$	$h(1) \approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$ $= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$ $= 3 - 1 + \frac{1}{2} - \frac{23}{48}$
In part (d) computing the magnitude of the first omitted term without communication about error as in $\frac{f^{(4)}(0)}{5!} = \frac{54}{120} = \frac{9}{20} = 0.45$; $\boxed{\frac{9}{20} = 0.45}$	The first term of the series for $h(1)$ omitted in the approximation is $\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$ $\lvert\text{Error in the approximation}\rvert \le \left\lvert\frac{9}{20}\right\rvert = 0.45$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

At a beginning level of understanding, students may discover that a computation produces an answer to a problem without knowing why or how that calculation produces the desired result. Advancing to a higher level of understanding, giving meaning to computation, is an important stage in learning a subject. An example of this is the alternating series error bound in part (d). When asked to show that an approximation differs from the target value by no more than $0.45,$ it is not just sufficient to produce a number that is 0.45 or less—somehow that number has to be tied to the error in the approximation. Many responses showed an awareness of the alternating series error bound as a number computed from the absolute value of the first term of the alternating series that is omitted by the approximating expression. But the relevance of that number and the term "error bound" was sometimes missed. Teachers can encourage students to not just crank out a number, but to communicate what that number means or "shows" in the given context.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- FRQ practice questions for teachers to use as formative assessment pieces are now available as part of the collection of new resources for teachers for the 2019 school year. These items begin with scaffolded questions that represent what students are ready for at the beginning of the school year and that continue on to present an increased challenge as teachers progress through the course. These resources are available on AP Classroom with the ability to search for specific question types and topics so that teachers are able to find the new collection of FRQ practice questions and the fully developed scoring guidelines that accompany each question.
- In part (a) some students might have benefited from instructional strategies designed to develop Skill 2.B: Identify mathematical information from graphical, numerical, analytical, and/or verbal representations. On p. 216 of the *AP Calculus AB and BC Course and Exam Description* (*CED*), see some key questions, sample activities, and sample instructional strategies designed to develop this skill.
- In part (c) some students did not use term-by-term integration. On p. 197 of the *CED*, see a link to a classroom resource to help students with Topic 10.15 content and skills: "Infinite Series."
- To develop student understanding of error bounds in Taylor polynomial approximation, consider using the classroom resource found linked in Topic 10.12 on p. 194 of the *CED*: "Approximation."