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Don't Forget the Differential Equations: Finishing 2005 BC4

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Don't Forget the Differential Equation: Finishing 2005 BC4

by Steve Greenfield*

August 19, 2006

Differential equations

Problem BC4 on the 2005 AP Calculus exam asked students to sketch a slope field and solution curve for the differential equation $y' = 2x - y$, investigate a critical point of this curve, compute the approximate value of the solution using Euler's method with two steps of equal size, and, finally, compare the approximate value to the actual value. A request to compute y'' was intended to help students by suggesting concavity considerations. The last part of the problem also required students to explain their reasoning about the comparison, and likely accounts for the unusually small percent of perfect scores on this problem. I focus in this note on the last part of the problem and begin with an analysis of a simpler but similar problem. Then the relevant sections of BC4 are reviewed with comments.

A familiar graph

The very nice curve $y = x^2$ shown in Figure 1 is always concave up. We are assured further of this because $y'' = 2 > 0$. This curve is also the graph of a solution to an initial value problem for a differential equation: $\begin{cases} y' = 2x \\ y(0) = 0 \end{cases}$.

All solutions of the differential equation $y' = 2x$ are of the form $f(x) = x^2 + C$ for various values of the constant, C. These are vertical translations, up and down, of the original parabola.

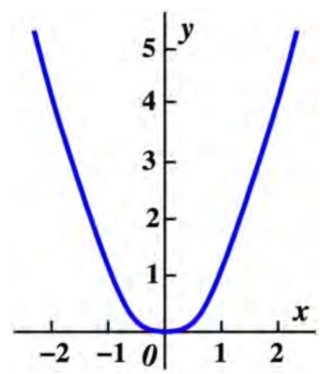

Figure 1

Euler steps

An Euler step of length Δx for the initial value problem $\begin{cases} y' = g(x,y) \\ y(x_0) = y_0 \end{cases}$ replaces (x_0, y_0) by $(x_0 + \Delta x, y_0 + g(x_0, y_0)\Delta x)$. This is a tangent line approximation to $y(x_0 + \Delta x)$. Using several steps of Euler's method to approximate the solution of $\begin{cases} y' = 2x \\ y(0) = 0 \end{cases}$ means repeatedly using tangent line approximations to $f(x) = x^2 + C$ for various values of C. Since all of these curves are concave

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up, the tangent line approximations will all be below the true value. Let's investigate this in more detail with several examples using different step sizes.

We start at $\left(\frac{2}{3}, \frac{4}{9}\right)$ on the curve $y = x^2$ and do an Euler step with $\Delta x = 1$. The differential equation is $y' = 2x$ so we will move from $\left(\frac{2}{3}, \frac{4}{9}\right)$ to $\left(\frac{2}{3} + 1, \frac{4}{9} + 2\left(\frac{2}{3}\right) 1\right)$. This point is $\left(\frac{5}{3}, \frac{16}{9}\right)$. This is certainly under the graph of $y = x^2$ because a tangent line to a concave up curve is below the curve. Notice that $\frac{16}{9} \approx 1.778 < \left(\frac{5}{3}\right)^2 \approx 2.778$.

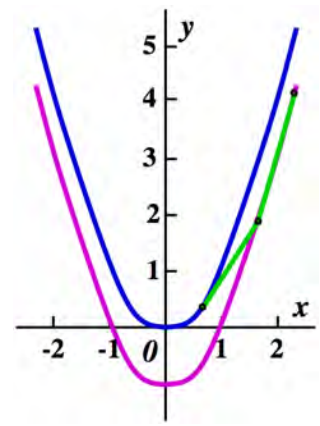


Figure 2

Let's compute another Euler step with $\Delta x = \frac{1}{2}$. We use the slope at the point $\left(\frac{5}{3}, \frac{16}{9}\right)$. If we find a solution to $y' = 2x$ which passes through $\left(\frac{5}{3}, \frac{16}{9}\right)$, the direction along the line tangent to that solution curve at $\left(\frac{5}{3}, \frac{16}{9}\right)$ is the direction of the next Euler step. The solution curve passing through $\left(\frac{5}{3}, \frac{16}{9}\right)$ is $y = x^2 - 1$. Now $y = x^2$ and $y = x^2 - 1$ are both solution curves to $y' = 2x$. Solution curves can't cross (see Appendix I), so if one of these curves is below the other at any one value of x, it must be below in all of its domain. But the tangent line approximation to $y = x^2 - 1$ is below $y = x^2 - 1$ since that curve is also concave up. Therefore (without computation!) the second Euler step's approximation must be below the true value on the first curve. Let's check. We must use the new slope, $2\left(\frac{5}{3}\right) = \frac{10}{3}$ and step size $\frac{1}{2}$ applied to the point $\left(\frac{5}{3}, \frac{16}{9}\right)$. The result is $\left(\frac{5}{3} + \frac{1}{2}, \frac{16}{9} + \left(\frac{10}{3}\right)\left(\frac{1}{2}\right)\right)$. This is $\left(\frac{13}{6}, \frac{31}{9}\right) \approx (2.167, 3.444)$. And $\left(\frac{13}{6}\right)^2 \approx 4.694$ which is considerably larger than 3.444.

Figure 2 shows an attempt to draw the graphs accurately. The two line segments illustrate our Euler steps of length 1 and $\frac{1}{2}$. The line segment corresponding to the second Euler step obscures much of the (rather flat) curve $y = x^2 - 1$ for x between $\frac{5}{3}$ and $\frac{13}{6}$.

Other differential equations

Our original curve, $y = x^2$, is a solution curve to many other differential equations. For example, $y = x^2$ solves $y' = 2x + \arctan(y - x^2)$ because $\arctan(0) = 0$ and $y = x^2$ implies that $y - x^2$ is always 0. Let's look at a differential equation that is less complicated.

Consider $y' = 2x + 10(y - x^2)$. This is not as easy to solve as our original differential equation, but explicit solutions can be found. We may use a trick (finding an *integrating factor*; please see Appendix II) to discover all of the exact solutions. These solutions are $f(x) = x^2 + Ce^{10x}$. Just as in the previous case, the constant, C, specifies an initial condition at 0: $f(0) = C$. When $C = 0$ we get the familiar parabola $y = x^2$ again.

Let's examine Euler's method applied to the solution of $\begin{cases} y' = 2x + 10(y - x^2) \\ y(0) = 0 \end{cases}$ and step *backwards* from 0 with $\Delta x = -\frac{1}{2}$. The solution curve is $y = x^2$ and the slope of this curve at $(0, 0)$ is 0. Our Euler step replaces $(0, 0)$ with $\left(-\frac{1}{2}, 0\right)$. Let's use another Δx step of $-\frac{1}{2}$. The slope we need to use is $2x + 10(y - x^2)$ when $x = -\frac{1}{2}$ and $y = 0$. This slope is $-1 + 10\Big(-\left(-\frac{1}{2}\right)^2\Big) = -\frac{7}{2}$. Therefore we go from $\left(-\frac{1}{2}, 0\right)$ to $\left(-\frac{1}{2} - \frac{1}{2}, 0 + \left(-\frac{7}{2}\right)\left(-\frac{1}{2}\right)\right)$ which is the point $\left(-1, \frac{7}{4}\right)$. This point is above $y = x^2$ because $\frac{7}{4} > 1 = (-1)^2$, so that two steps of Euler's method now gives us something which is *greater* than the true value of the solution.

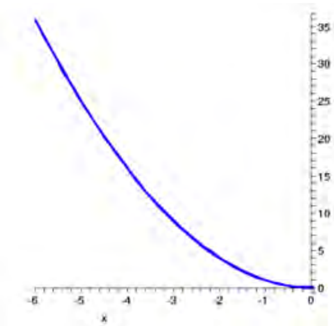


Figure 3

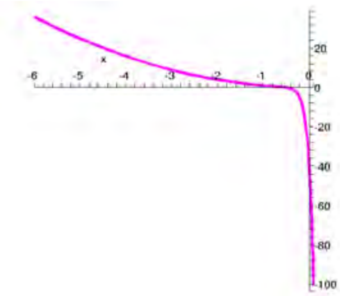


Figure 4

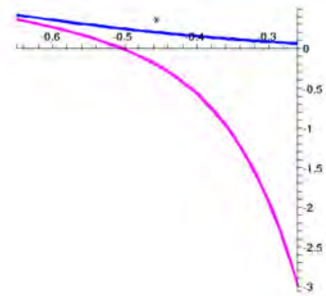


Figure 5

This may seem complicated and perhaps not intuitive. Above are some graphs created by `Maple`. Figure 3 shows $y = x^2$ for $-6 \le x \le .1$. Figure 4 shows the solution curve $y = x^2 + Qe^{10x}$ where $Q = -\frac{1}{4}e^5 \approx -37.103$, again for $-6 \le x \le .1$. This solution curve also satisfies $y' = 2x + 10(y - x^2)$ and passes through the point $\left(-\frac{1}{2}, 0\right)$. Figure 5 shows both curves for $-.65 \le x \le -.25$. Notice that the three "windows" are all very different (even though Figures 3 and 4 have the same horizontal "size", y goes from 0 to 36 in Figure 3 and from -100 to 36 in Figure 4).

Some geometry: the changing shape of solution curves

The solution curve $y = x^2 + Qe^{10x}$ with the value of Q specified above has second derivative $2 + 100Qe^{10x}$. The second derivative changes sign at $x = \frac{1}{10}\ln\left(-\frac{1}{50Q}\right) = \frac{1}{10}\ln\left(\frac{2}{25e^5}\right) \approx -.753$ and therefore the solution curve has an inflection point at (approximately!) $(-.753, .546)$.

The first Euler step moves horizontally to the left. The second Euler step is tangent to the lower solution curve in the region where that solution curve is concave down. The tangent line is therefore above the lower solution curve, and the slope of the tangent line is also sufficiently negative so that an Euler step with $\Delta x = -\frac{1}{2}$ ends above $y = x^2$.

Figure 6 shows the two curves together with line segments to display the two Euler steps calculated.

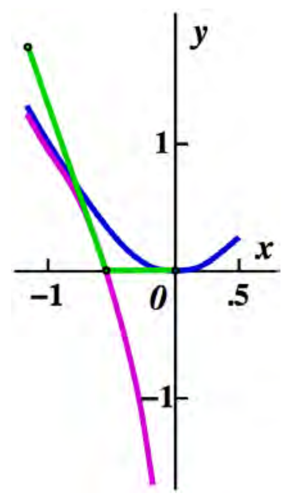


Figure 6

Even for $y = x^2$ . . .

You can't forget the differential equation. The solution curves for the two initial value problems

$$\begin{cases} y' = 2x \\ y(0) = 0 \end{cases} \qquad \begin{cases} y' = 2x + 10(y - x^2) \\ y(0) = 0 \end{cases}$$

are the same: $y = x^2$. For the first problem, two Euler steps with $\Delta x = -\frac{1}{2}$ starting at $(0, 0)$ give an approximate value of $\frac{1}{2}$, less than the true value of the solution, which is 1. For the second problem, two Euler steps with $\Delta x = -\frac{1}{2}$ starting at $(0, 0)$ give an approximate value of $\frac{7}{4}$, more than the true value of the solution, which is 1.

And now for the last point on 2005 BC4

BC4 asked about the solution to $\begin{cases} y' = 2x - y \\ y(0) = 1 \end{cases}$. The problem first requested that students draw a slope field at a dozen points, and then students were asked to sketch the solution curve through $(0, 1)$. Figure 7 shows a correct answer.

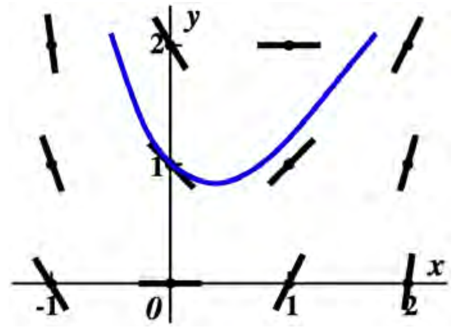


Figure 7

BC4 did *not* ask for an exact solution, which can be found. It is $f(x) = 2x - 2 + 3e^{-x}$. Certainly some students discovered this formula, but the time and energy needed almost surely reduced their success in the problem!

Part (c) of BC4 requested that the student apply two Euler steps with $\Delta x = -.2$ and report the resulting approximate value of $f(-.4)$. This approximation is 1.52, and it is less than the exact value, which is about 1.657.

Part (d) first asked students to find y'' in terms of x and y. A mild use of the Chain Rule has $y' = 2x - y$ yield $y'' = 2 - y' = 2 - (2x - y) = 2 - 2x + y$. The second point would be earned for part (d) if students could explain why the approximation found in (c) was indeed less than the true value of $f(-.4)$. The first Euler step is to the left because $\Delta x = -.2$ is negative. The step's end must be under the original solution curve $y = f(x)$ because that curve is concave up (see below for information about the sign of y''). Another Euler step will need information that can't be gotten from the original solution curve. That information comes from the slope of an *unknown* solution curve which passes through the point at the end of the first Euler step, somewhere in the region indicated in Figure 8. If we knew that this curve was always concave up in that region, we then could conclude that its tangent line was underneath, and therefore the second step's approximation would be less than the true value.

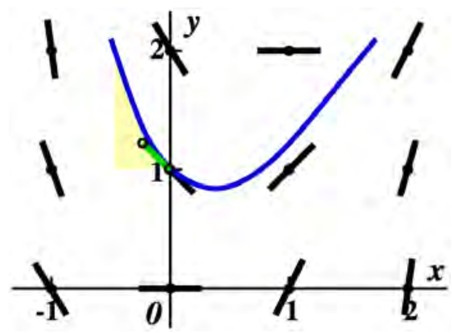


Figure 8

The second derivative of any solution curve of the differential equation $y' = 2x - y$ is positive in the region indicated. There we know that y is positive and x is negative because the region is part of the second quadrant. So $y'' = 2 - 2x + y > 0$ there. The concave up behavior of the original solution curve and the unknown solution curve is guaranteed where we need it.

If we did not know enough about the shape of the lower solution curve, we could not decide whether the approximation was larger or smaller than the true value of $f(-.4)$. Figure 9 is a close-up view of several hypothetical solution curves. The original curve through $(0, 1)$ is certainly concave up in this window. The first Euler step sits nicely below it. The lower solution curve is actually concave up for x's near $-.2 \leq x \leq 0$ *and* for x's near $-.4$. The lower solution curve is concave down for x's near the interval $-.35 \leq x \leq -.25$. This concave down bump displayed allows the second Euler step to finish above $y = f(x)$ when $x = -.4$. To be sure that the second Euler step ends below $f(-.4)$ we must know that solution curves are concave up in an entire region containing $-.4 \leq x \leq 0$.

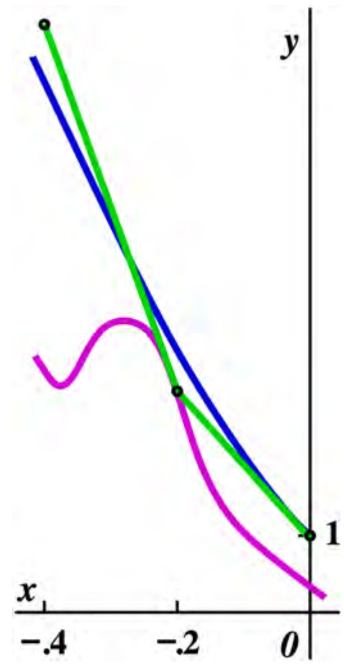


Figure 9

So the second Euler step may be above $f(-.4)$ if we allow the solution curves to *change their shapes*. I think that this phenomenon is very subtle.

Appendix I: Solutions of differential equations

Here is one version of a wonderful theorem which solves (?) all of your problems involving differential equations:

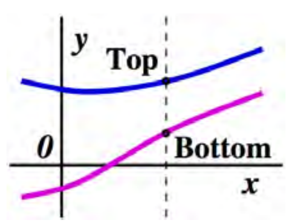
> **Suppose that $f(x, y)$ is a differentiable function of x and y and (x_0, y_0) is in the domain of $f(x, y)$. The initial value problem $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$ has exactly one solution whose domain is an open interval containing x_0. If x is any number in the solution's domain and y is the corresponding value of the solution, $f(x, y)$ must be defined.**

This result doesn't help very much with examples.

• The theorem declares there *is* a solution, but the solution may not be recognizable in terms of standard functions. The initial value problem $\begin{cases} y' = e^{x^2} \\ y(0) = 0 \end{cases}$ has the solution $y(x) = \int_0^x e^{t^2}\,dt$ with no real simplifications possible. And I don't know any simple way, even using integrals, to write the solution to the initial value problem $\begin{cases} y' = 2x + \arctan(y - x^2) \\ y(0) = 1 \end{cases}$ which must exist according to the theorem.

• The theorem doesn't guarantee that the solution will live very long. Take $f(x, y) = xy^2$, a very nice polynomial, defined and differentiable for all (x, y)'s. The differential equation $y' = xy^2$ is separable, and has solutions $y = \frac{2}{C - x^2}$ for $C \neq 0$. The initial condition this satisfies is $x_0 = 0$ and $y_0 = \frac{2}{C}$. The solution through $(0, 0)$ is the entire x-axis. For positive C, the domain of the solution curve is only $-\sqrt{C} < x < \sqrt{C}$, and as $C \to 0^+$ (so the initial condition in y goes to $+\infty$), this domain shrinks to a point. Specifically, the initial value problem $\begin{cases} y' = xy^2 \\ y(0) = 200 \end{cases}$ has solution $y(x) = \frac{2}{\frac{1}{100} - x^2}$, whose domain is only $-\frac{1}{10} < x < \frac{1}{10}$. This tiny domain is not evident when examining xy^2.

• There is one nice consequence. Because the initial value problems considered here have exactly *one* solution, distinct solution curves can't cross. If the curves did cross, then they would both satisfy the same differential equation and the same initial value condition at the crossing point. The theorem declares ("exactly one") that the curves must be *identical*, a contradiction. So if two distinct solution curves are both defined on the same interval



with different initial conditions, the curve "on top" must stay on top because otherwise (the Intermediate Value Theorem!) the curves would intersect and would have to be identical.

This pretty picture comes with a warning, though. Distinct solution curves to higher order differential equations may indeed cross. Students may encounter second order differential equations coming from velocity/acceleration problems with solution curves that cross. For example, $y = x$ and $y = 2x$ are both solutions of $\begin{cases} y'' = 0 \\ y(0) = 0 \end{cases}$. The geometry of solution curves to higher order differential equations is more complicated.

Appendix II: Integrating factors

Two of the differential equations discussed here are $y' = 2x + 10(y - x^2)$ and $y' = 2x - y$. Both are examples of *first-order linear differential equations* and both can be solved in terms of standard functions by using an integrating factor.

Suppose we want to solve the differential equation $y' = A(x)y + B(x)$. Rearrange it as $y' - A(x)y = B(x)$. Now let's *guess* an antiderivative of $-A(x)$. That is, suppose we know a function $C(x)$ so that $C'(x) = -A(x)$. Then multiply the rearranged equation by $e^{C(x)}$ to get $e^{C(x)}y' - A(x)e^{C(x)}y = e^{C(x)}B(x)$. The left-hand side of this equation is now exactly a derivative: $\frac{d}{dx}\left(e^{C(x)}y\right) = e^{C(x)}y' + e^{C(x)}C'(x)y = e^{C(x)}y' - A(x)e^{C(x)}y$. So we can (try to) antidifferentiate the whole equation.

Describing this method abstractly doesn't help me very much. I hope some examples will help. Let's look at the two equations mentioned earlier.

How to solve $y' = 2x - y$

The differential equation $y' = 2x - y$ becomes $y' + y = 2x$. Here $-A(x)$ is 1 and $C(x)$ is x so we multiply by e^x and get $e^x y' + e^x y = 2xe^x$. The left-hand side is the derivative of $e^x y$ and the right-hand side is the derivative of $2xe^x - 2e^x$ (integrate by parts). For the general solution, add a constant to the right-hand side, and get $e^x y = 2xe^x - 2e^x + C$. Thus $y = 2x - 2 + Ce^{-x}$ is the general solution to $y' = 2x - y$. If we want the solution to go through the point $(0, 1)$, substitute $x = 0$ and $y = 1$ to learn that C must be 3.

How to solve $y' = 2x + 10(y - x^2)$

The differential equation $y' = 2x + 10(y - x^2)$ becomes $y' - 10y = 2x - 10x^2$. Now $-A(x)$ is -10 and $C(x)$ is $-10x$ so here we multiply by e^{-10x}. The result is $e^{-10x}y' - 10e^{-10x}y = (2x - 10x^2)e^{-10x}$. We did this so that the left-hand side would be a derivative of something, and that something is $e^{-10x}y$. Now we must antidifferentiate $(2x - 10x^2)e^{-10x}$. We're lucky because this is the derivative of x^2e^{-10x} and I don't need to struggle with two integrations by parts. Therefore $e^{-10x}y = x^2e^{-10x} + C$ so that $y = x^2 + Ce^{10x}$ as was stated earlier.

But be careful!

Some calculus books include examples resembling the two shown above, usually after material on Newton's law of cooling. But the integrating factor technique rarely succeeds in finding solutions involving standard functions for more complicated equations. The differential equation $y' = 7 - 2xy$ looks as simple as the two earlier examples. The needed integrating factor is an antiderivative of e^{x^2}. Again, this can't be expressed in terms of standard elementary functions.