Approximating Limit Values from a Graph

Plan

Learning Goals

Students will be able to:

- Recognize limit statements that correspond to vertical and horizontal asymptotes.
- Write verbal descriptions that correspond to symbolic limit statements.

There are many contexts in calculus involving approximation— that is, finding a value or function that is nearly but not exactly correct. In some cases, the value cannot be known and so approximation is the only option. In other cases, approximation is used as a guide for determining whether the more precise calculation is reasonable.

This lesson builds the skill of approximation by having students connect the graphical behaviors of functions with corresponding limit expressions.

Student misunderstandings

- Students confuse the function value at c with behavior as x approaches c – this is particularly true when there's a visual cue in the graph that prompts the student to select the function value rather than the limit value); emphasize that the function value is irrelevant when estimating the limit.
- The student may confuse notation such as "$x \to 2^-$" thinking that x is taking on only negative values, rather than x is "being approached from the negative side of the graph."
- The student may struggle to determine which particular graphical behavior is associated with a given limit statement.
- Students struggle to appropriately estimate the value of a limit for non-integer values.

Materials

The following supplies are needed:

- student activity sheets (1 per student)
- pre-cut cards separated into three stacks: 8 graph cards, 8 limit cards, and 8 verbal description cards, located on the last three pages of this document. (1 set per small group – see end of document for printable versions)

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AP CALCULUS STUDENT HANDOUT

Associating Limits with Graphical Behavior

You have in front of you three stacks of cards:

- One stack has cards showing eight different **graphs** of a function $f(x)$ with a specific region on the graph circled.
- Another stack has cards showing eight different **limit statements** in symbolic notation.
- The third stack has cards showing eight different **verbal descriptions** of graphical behavior.

Your task is to rearrange the cards into eight stacks, each containing a graph card, a limit card, and a verbal description card that match one another. Once you have finished this task, record your results in the following table, including the answer for each limit statement. If the limit is a finite number, write that number. If the function is unbounded in the positive or negative direction, write $+\infty$ or $-\infty$, respectively. Otherwise, write DNE for "does not exist." The first row of the table is filled out as an example.

Graphical Region	Limit Statement	Verbal Description	Answer
1	H	VI	$-\infty$
2	Answer: E	Answer: I	Answer: 4
3	Answer: A	Answer: VII	Answer: 5
4	Answer: F	Answer: IV	Answer: DNE
5	Answer: G	Answer: V	Answer: 1
6	Answer: B	Answer: VIII	Answer: 6
7	Answer: C	Answer: II	Answer: -1
8	Answer: D	Answer: III	Answer: DNE

Now share your findings with a partner and resolve any differences you have by discussing with your partner, and, if necessary, other members of the class.

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Teach

Engage

This is a think-pair-share activity. Have students work with a partner to complete the activity, then share out.

Note that DNE means "does not exist." Most, but not all, textbooks consider functions whose y-coordinates are unbounded in either the positive or negative direction to have limits that do not exist (see Graph 1, for instance). For this and subsequent activities in this lesson, students are encouraged to use the more descriptive $+\infty$ or $-\infty$ for the limits of such functions.

This activity is a quick check to see that students are able to correctly read limits from a graph. As students check their answers with each other and resolve any differences, the teacher is able to determine which students need additional explanations in order to be able to associate limit statements with appropriate regions of the graph.

AP CALCULUS STUDENT HANDOUT

The Connection Between Asymptotes and Limits

Part I: For each question, describe what the given information tells you about the graph of $y = f(x)$. Then, decide whether the given information allows you to identify a horizontal or vertical asymptote for the graph of $y = f(x)$. If there is sufficient information, state the equations of any asymptotes.

Example: $\lim_{x \to 2} f(x) = \infty$

- Explain what this tells you about the graph of $y = f(x)$
  As x gets closer to 2, both from the left and from the right, the y-coordinates are unbounded, getting larger and larger in the positive direction.
- Vertical asymptote(s)? ☐ No ☑ Yes, equation(s): ___**x = 2**_____
- Horizontal asymptote(s)? ☑ No ☐ Yes, equation(s): _____

1. $\lim_{x \to 3^-} f(x) = +\infty$, $\lim_{x \to 3^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x gets closer to 3 from the left, the y-coordinates are unbounded, getting larger and larger in the positive direction. As x gets closer to 3 from the right, the y-coordinates are unbounded in the negative direction.
- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): _x = 3_____
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

2. $\lim_{x \to +\infty} f(x) = 2$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x moves farther and farther to the right, the y-coordinates get closer and closer to 2.
- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): _y = 2_____

3. $\lim_{x \to -1^-} f(x) = 4$, $\lim_{x \to -1^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x approaches -1 from the left, the y-coordinates get closer and closer to 4. As x approaches -1 from the right, the y-coordinates are unbounded in the negative direction.
- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): _x = -1_____
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

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Guided Practice

Read through the directions and example together, then have students work on #1-7 and check their answers with a partner. Go over any questions at the end of the exercise.

4. $\lim_{x \to +\infty} f(x) = 2, \lim_{x \to -\infty} f(x) = 4$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x moves farther and farther to the right, the y-coordinates get closer and closer to 2. As x moves farther and farther to the left, the y-coordinates get closer and closer to 4.
- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): $y = 2,\ y = 4$
- Horizontal asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): _____

5. $\lim_{x \to 4^-} f(x) = 2, \lim_{x \to 4^+} f(x) = 3$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x approaches 4 from the left, the y-coordinates get closer and closer to 2. As x approaches 4 from the right, the y-coordinates get closer and closer to 3.
- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

6. $\lim_{x \to 1} f(x) = +\infty, f(1) = 4$.

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x gets closer and closer to 1, both from the left and from the right, the y-coordinates become larger and larger in the positive direction. Also, there is a point on the vertical asymptote of the graph at (1, 4).
- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): $x = 1$
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

7. $\lim_{x \to 2^+} f(x) = -\infty, \lim_{x \to 3^-} f(x) = +\infty, \lim_{x \to +\infty} f(x) = +\infty, \lim_{x \to -\infty} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$
  Answer: As x gets closer to 2 from the right, the y-coordinates are unbounded in the negative direction. As x gets closer to 3 from the left, the y-coordinates are unbounded in the positive direction. As x moves farther to the right, the y-coordinates are unbounded in the positive direction. As x moves farther to the left, the y-coordinates are unbounded in the negative direction.
- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): $x = 2,\ x = 3$
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

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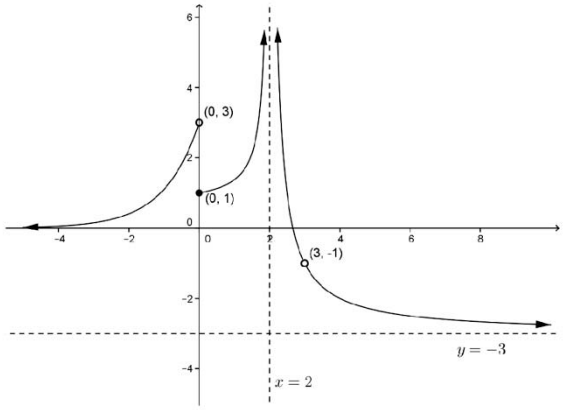
AP CALCULUS STUDENT HANDOUT

Part II: This part of the activity concentrates on writing correct notation for limit statements as well as making the connection between limits and graphical behavior. Shown below is the graph of a function $f(x)$. There are eleven limit statements, including one-sided and two-sided limits, based on the labeled points and lines on this graph. Write at least ten of these limit statements.



Answers:

1. $\lim_{x \to -\infty} f(x) = 0$
2. $\lim_{x \to \infty} f(x) = -3$
3. $\lim_{x \to 0^-} f(x) = 3$
4. $\lim_{x \to 0^+} f(x) = 1$
5. $\lim_{x \to 0} f(x)$ DNE
6. $\lim_{x \to 2^-} f(x) = +\infty$
7. $\lim_{x \to 2^+} f(x) = +\infty$
8. $\lim_{x \to 2} f(x) = +\infty$
9. $\lim_{x \to 3^-} f(x) = -1$
10. $\lim_{x \to 3^+} f(x) = -1$
11. $\lim_{x \to 3} f(x) = -1$

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Independent Practice

Students work individually on writing limit statements for the graph provided. If time does not permit for them to complete the assignment in class, this could be given as homework.

Make certain students are using correct notation. Common errors are to omit the name of the function or to insert an extra equals sign between "lim" and "$f(x)$".

Assess

Direct students to take the Focus Quiz.

Additional Learning Resources

- More practice of reading limits from a graph can be obtained by watching this video and then completing the associated practice.
  https://www.khanacademy.org/math/differential-calculus/limits_topic/calculus-estimating-limits-graph/v/determining-which-limit-statements-are-true

Student handouts with answers:

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The Connection Between Asymptotes and Limits

Part I: For each question, describe what the given information tells you about the graph of $y = f(x)$. Then, decide whether the given information allows you to identify a horizontal or vertical asymptote for the graph of $y = f(x)$. If there is sufficient information, state the equations of any asymptotes.

Example: $\lim_{x \to 2} f(x) = \infty$

- Explain what this tells you about the graph of $y = f(x)$

  As x gets closer to 2, both from the left and from the right, the y-coordinates are unbounded, getting larger and larger in the positive direction.

- Vertical asymptote(s)? ☐ No ☑ Yes, equation(s): **$x = 2$**
- Horizontal asymptote(s)? ☑ No ☐ Yes, equation(s): _____

1. $\lim_{x \to 3^-} f(x) = +\infty, \lim_{x \to 3^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As x gets closer to 3 from the left, the y-coordinates are unbounded, getting larger and larger in the positive direction. As x gets closer to 3 from the right, the y-coordinates are unbounded in the negative direction.

- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): $x = 3$
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

2. $\lim_{x \to +\infty} f(x) = 2$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As x moves farther and farther to the right, the y-coordinates get closer and closer to 2.

- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): $y = 2$

3. $\lim_{x \to -1^-} f(x) = 4, \lim_{x \to -1^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As x approaches -1 from the left, the y-coordinates get closer and closer to 4. As x approaches -1 from the right, the y-coordinates are unbounded in the negative direction.

- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): $x = -1$
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

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4. $\lim_{x\to+\infty} f(x) = 2, \lim_{x\to-\infty} f(x) = 4$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As *x* moves farther and farther to the right, the *y*-coordinates get closer and closer to 2. As *x* moves farther and farther to the left, the *y*-coordinates get closer and closer to 4.

- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): *y = 2, y = 4*
- Horizontal asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): _____

5. $\lim_{x\to4^-} f(x) = 2, \lim_{x\to4^+} f(x) = 3$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As *x* approaches 4 from the left, the *y*-coordinates get closer and closer to 2. As *x* approaches 4 from the right, the *y*-coordinates get closer and closer to 3.

- Vertical asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

6. $\lim_{x\to1} f(x) = +\infty, f(1) = 4$.

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As *x* gets closer and closer to 1, both from the left and from the right, the *y*-coordinates become larger and larger in the positive direction. Also, there is a point on the vertical asymptote of the graph at (1, 4).

- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): *x = 1*
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

7. $\lim_{x\to2^+} f(x) = -\infty, \lim_{x\to3^-} f(x) = +\infty, \lim_{x\to+\infty} f(x) = +\infty, \lim_{x\to-\infty} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

  Answer: As *x* gets closer to 2 from the right, the *y*-coordinates are unbounded in the negative direction. As *x* gets closer to 3 from the left, the *y*-coordinates are unbounded in the positive direction. As *x* moves farther to the right, the *y*-coordinates are unbounded in the positive direction. As *x* moves farther to the left, the *y*-coordinates are unbounded in the negative direction.

- Vertical asymptote(s)? ☐ No Answer: ☑ Yes, equation(s): *x = 2, x = 3*
- Horizontal asymptote(s)? Answer: ☑ No ☐ Yes, equation(s): _____

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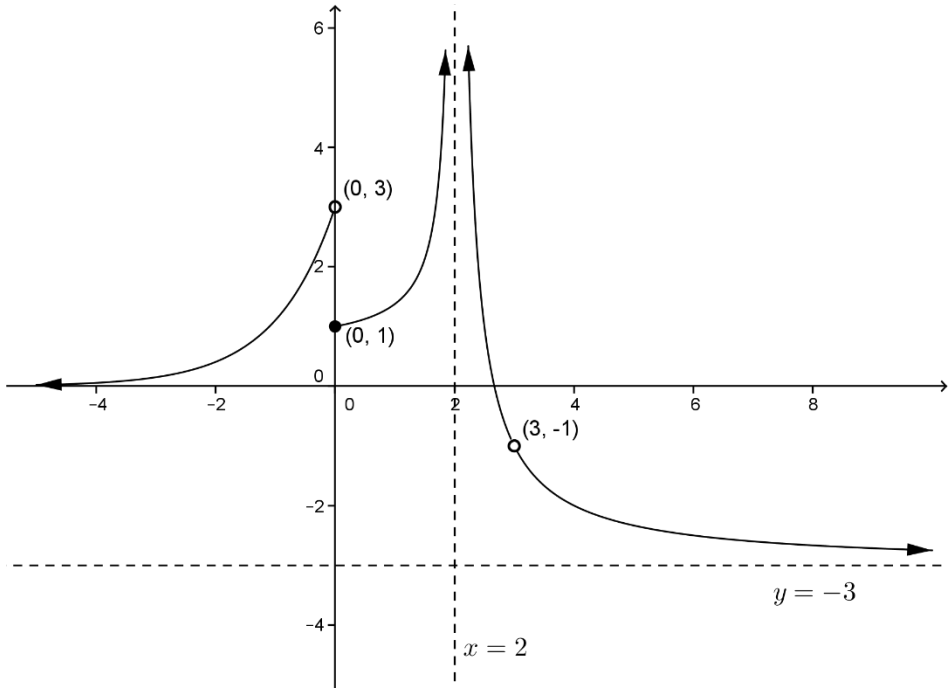
Part II: This part of the activity concentrates on writing correct notation for limit statements as well as making the connection between limits and graphical behavior. Shown below is the graph of a function $f(x)$. There are eleven limit statements, including one-sided and two-sided limits, based on the labeled points and lines on this graph. Write at least ten of these limit statements.



Answers:

1. $\lim_{x \to -\infty} f(x) = 0$
2. $\lim_{x \to \infty} f(x) = -3$
3. $\lim_{x \to 0^-} f(x) = 3$
4. $\lim_{x \to 0^+} f(x) = 1$
5. $\lim_{x \to 0} f(x)$ DNE
6. $\lim_{x \to 2^-} f(x) = +\infty$
7. $\lim_{x \to 2^+} f(x) = +\infty$
8. $\lim_{x \to 2} f(x) = +\infty$
9. $\lim_{x \to 3^-} f(x) = -1$
10. $\lim_{x \to 3^+} f(x) = -1$
11. $\lim_{x \to 3} f(x) = -1$

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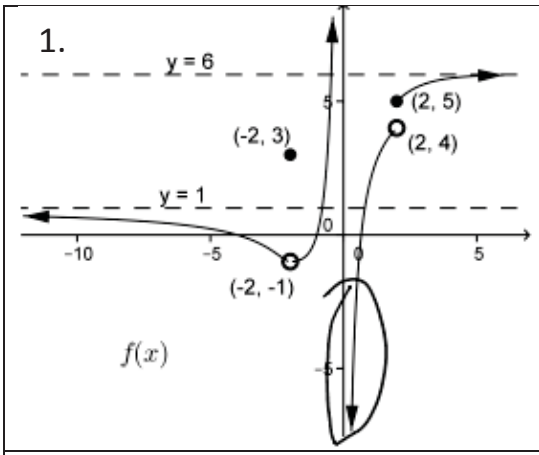
1.



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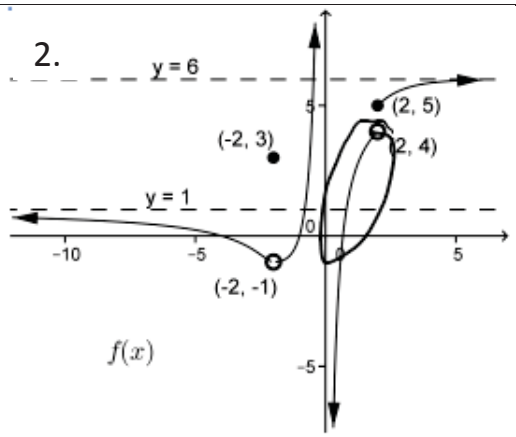

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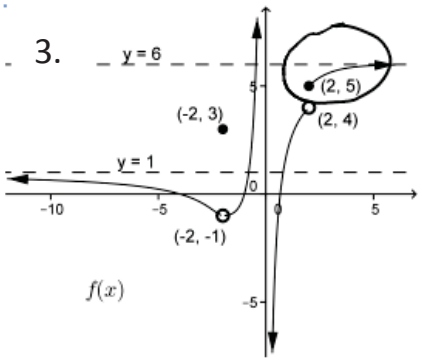

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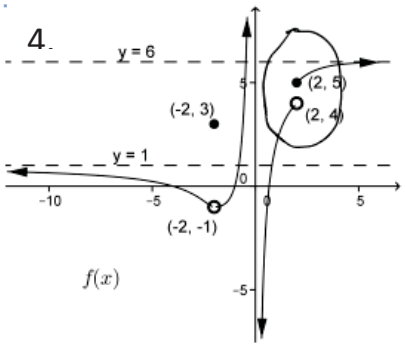

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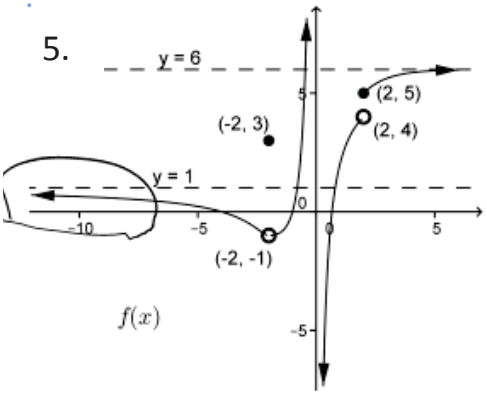

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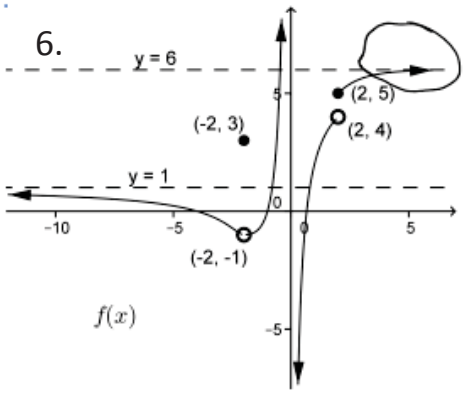

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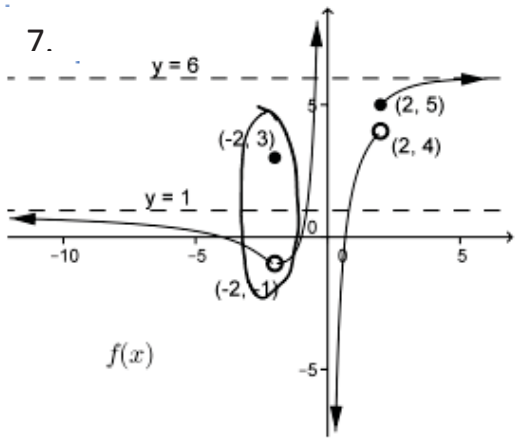

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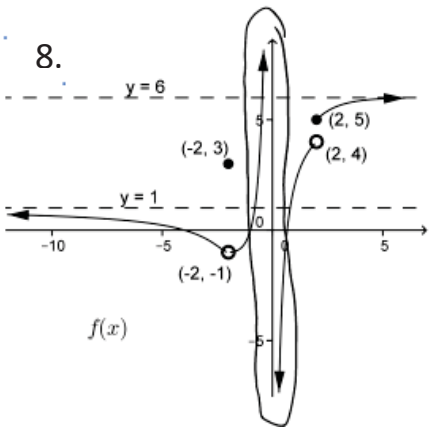

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A. $\lim_{x \to 2^+} f(x)$	B. $\lim_{x \to \infty} f(x)$
C. $\lim_{x \to -2} f(x)$	D. $\lim_{x \to 0} f(x)$
E. $\lim_{x \to 2^-} f(x)$	F. $\lim_{x \to 2} f(x)$
G. $\lim_{x \to -\infty} f(x)$	H. $\lim_{x \to 0^+} f(x)$

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I. As the values of x get closer and closer to 2 from the left-hand side (that is, values like 1.9, 1.99, etc.), this is what the values of $f(x)$ are approaching.	II. As the values of x get closer and closer to -2, both from the left-hand side and the right-hand side, this is what the values of $f(x)$ are approaching.
III. As the values of x get closer and closer to -2, both from the left-hand side and the right-hand side, this is what the values of $f(x)$ are approaching.	IV. As the values of x get closer and closer to 2, both from the left-hand side and the right-hand side, this is what the values of $f(x)$ are approaching.
V. As the values of x move farther and farther to the left, this is what the values of $f(x)$ are approaching.	VI. As the values of x get closer and closer to 0 from the right-hand side (that is, values like 0.1, 0.01, etc.), this is what the values of $f(x)$ are approaching.
VII. As the values of x get closer and closer to 2 from the right-hand side (that is, values like 2.1, 2.01, etc.), this is what the values of $f(x)$ are approaching.	VIII. As the values of x move farther and farther to the right, this is what the values of $f(x)$ are approaching.

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Example: $\lim_{x \to 2} f(x) = \infty$

- Explain what this tells you about the graph of $y = f(x)$

  As x gets closer to 2, both from the left and from the right, the y-coordinates are unbounded, getting larger and larger in the positive direction.

- Vertical asymptote(s)? ☐ No ☑ Yes, equation(s): ___**x = 2**_____
- Horizontal asymptote(s)? ☑ No ☐ Yes, equation(s): _____

1. $\lim_{x \to 3^-} f(x) = +\infty, \lim_{x \to 3^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

2. $\lim_{x \to +\infty} f(x) = 2$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

3. $\lim_{x \to -1^-} f(x) = 4, \lim_{x \to -1^+} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

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4. $\lim_{x\to+\infty} f(x) = 2, \lim_{x\to-\infty} f(x) = 4$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

5. $\lim_{x\to4^-} f(x) = 2, \lim_{x\to4^+} f(x) = 3$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

6. $\lim_{x\to1} f(x) = +\infty, f(1) = 4$.

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

7. $\lim_{x\to2^+} f(x) = -\infty, \lim_{x\to3^-} f(x) = +\infty, \lim_{x\to+\infty} f(x) = +\infty, \lim_{x\to-\infty} f(x) = -\infty$

- Explain what this tells you about the graph of $y = f(x)$

- Vertical asymptote(s)? ☐ No ☐ Yes, equation(s): _____
- Horizontal asymptote(s)? ☐ No ☐ Yes, equation(s): _____

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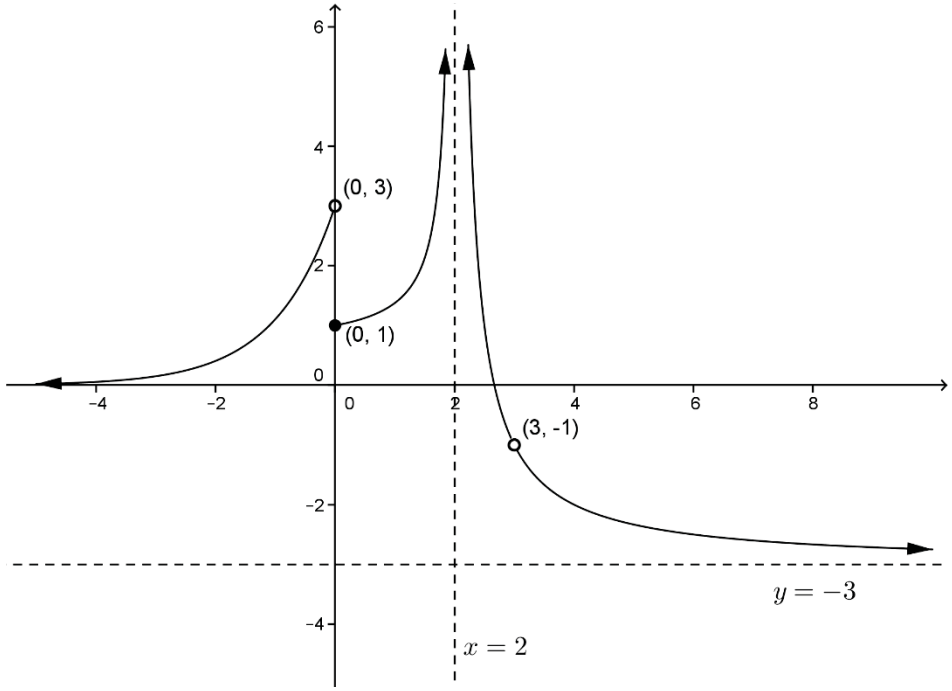
Part II: This part of the activity concentrates on writing correct notation for limit statements as well as making the connection between limits and graphical behavior. Shown below is the graph of a function $f(x)$. There are eleven limit statements, including one-sided and two-sided limits, based on the labeled points and lines on this graph. Write at least ten of these limit statements.



1._____ 6._____

2._____ 7._____

3._____ 8._____

4._____ 9._____

5._____ 10._____

11._____

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