# Exploring the FTC from Numerical and Graphical Points of View with the TI-89 

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In this activity, you will explore the Fundamental Theorem of Calculus from numeric and graphic perspectives. The exploration will give you additional practice with functions of the form $F(x)=\int_{a}^{x} f(t) d t$.

Define $\mathbf{y} \mathbf{1}$ and $\mathrm{y}^{2}$ in the $\mathbf{y}=$ editor, and set up the viewing window as shown.


Set your calculator to radian mode. Note that $\mathrm{y}^{2}$ is a function defined as a definite integral of $\mathbf{y} 1$. That is, $\mathbf{y}^{2}(\mathbf{x})=\int_{0}^{x} \cos \left(\frac{t^{2}}{2}\right) d t$.

Make sure that $\mathbf{y} \mathbf{1}$ is the only function selected for graphing, and look at the graph. Your graph should look like the one shown below.


If your graph is different, check that you've entered y 1 correctly, that your window is correct, and that you are in radian mode.

Now, you will create three lists. The first list, inputs, consists of $x$ values from the graph screen. These will serve as inputs into $\mathbf{y} 1$ and $\mathrm{y}^{2}$. To create inputs, follow these steps:

- Select seq ( from the CATALOG menu, as shown in the screen shot below.
- Fill in the arguments as shown below.
- The STO $\boldsymbol{C o m m a n d}$ is on the keypad, directly above the ON key.
- Use alpha-lock to enter inputs.


Now you will evaluate $y_{1}$ at each of the inputs in inputs and store the results in another list, ylouts. Edit the seq ( command as shown below.


Then evaluate $y^{2}$ at each of the inputs in inputs, and store the results in list int0outs. Again, edit the seq ( command as shown below.


Be prepared to wait a few minutes for all the integrals to be calculated. You will examine these values in a table format and use them in calculations later on.

Each value in int0outs results from the evaluation of a definite integral. For example, when 0.1 , the second value in inputs, is plugged into $y^{2}$, the value of

$$
\int_{0}^{0.1} \cos \left(\frac{t^{2}}{2}\right) d t=0.09999975
$$

results, which is the second value in int0outs.

1. Explain why the first value in intOouts, which results from evaluating $\mathrm{y}^{2}(0)$, is 0 .

Look at the graph of the integrand, $\mathbf{y} \mathbf{1}(\mathbf{x})$. Use the graph to answer questions 2 and 3 .
2. Why is the second value in int0outs greater than 0 ?
3. Why is the third value in int0outs, which represents $\int_{0}^{0.2} \cos \left(\frac{t^{2}}{2}\right) d t$, greater than the second value in int0outs, which represents $\int_{0}^{0.1} \cos \left(\frac{t^{2}}{2}\right) d t$ ?

Select 6:Data/Matrix Editor from the Apps menu and 3:New to make a new data set. Type ftc for the name of the Variable, and press ENTER.


Highlight c1, press ENTER, and type inputs, then press ENTER. This will put the values in the inputs list into c1. Put y1outs into c2, and put int0outs into c3. See the screen shots.


Remember that $\mathbf{c} 1$ contains $x$ values, $\mathbf{c} 2$ contains the values of the integrand, $\cos \left(\frac{t^{2}}{2}\right)$, and $\mathbf{c} 3$ contains the values of the integral.
4. From the list of data values stored in c3, for what value of $\mathbf{x}$ (stored in $\mathbf{c} 1$ ) does $\mathbf{y}^{2}(\mathbf{x})=\int_{0}^{x} \cos \left(\frac{t^{2}}{2}\right) d t$ appear to reach its first local maximum?
5. Explain why the twentieth value of c 3 is smaller than the nineteenth value of c 3 . That is, why is $\int_{0}^{1.9} \cos \left(\frac{t^{2}}{2}\right) d t<\int_{0}^{1.8} \cos \left(\frac{t^{2}}{2}\right) d t$ ?
6. From the list of data values stored in $\mathbf{c} 3$, for what positive value of $\mathbf{x}$ does $\mathbf{y}^{2}(\mathbf{x})=\int_{0}^{x} \cos \left(\frac{t^{2}}{2}\right) d t$ appear to reach its first local minimum?

Press F2: Plot Setup from the Data Editor and then F1:Define to set up Plot1 as a scatterplot of c3 versus c1 as shown in the screen shot below. Select Plot1 and y1 for graphing.

7. Press $\mathbf{F 5}$ and select $\mathbf{2}$ : Zero from the Math menu of the graph screen. Find the two smallest positive $x$-intercepts on the graph of $\mathbf{y} \mathbf{1}$, indicated by $a$ and $b$ in the screen shot above.
(a) $\qquad$ and (b) $\qquad$
8. Compare the answers to 7 (a) and 7 (b) with the answers to numbers 4 and 6 respectively. Explain why they are similar.
9. Find the $x$-intercept of the graph of $\mathbf{y} \mathbf{1}(\mathbf{x})$ that is closest to $\mathbf{x}=4$. $\qquad$
10. What happens with $\mathbf{y}^{2}(\mathbf{x})$ at the point found in number 9 ? $\qquad$
(Verify your answer numerically by looking at the table of values for $\mathbf{y} 2(\mathbf{x})$ in $\mathbf{c 3}$. Select 6: Data/Matrix Editor from the Apps menu and 1:Current to return to the data editor.)

Go back to the $\mathbf{y}=$ editor and define $\mathrm{y} 3=\int(\mathrm{y} 1(\mathrm{t}), \mathrm{t}, \mathbf{1}, \mathbf{x})$, thus changing the lower limit of integration from 0 to 1 . So we have $\mathrm{y} 3(\mathbf{x})=\int_{1}^{x} \cos \left(\frac{t^{2}}{2}\right) d t$. On the HOME screen, use the seq ( command to evaluate $y 3$ at all the inputs in inputs, and store the results in int1outs. Again, this command will take several minutes. Then return to the data editor, and put int1outs into c4. See the screen shots below.


Define Plot2 as a scatterplot of c4 versus c1, using the Plus symbol as the Mark. Select Plot1, Plot2, and $\mathbf{y} 1$ for graphing. Study the graph and table below.


Note that the first element of c4 represents y3 $(0)=\int_{1}^{0} \cos \left(\frac{t^{2}}{2}\right) d t=-\int_{0}^{1} \cos \left(\frac{t^{2}}{2}\right) d t$.
11. Why is the graph of $\mathbf{c} 4$ versus $\mathbf{c 1}$ (Plot2) below the graph of c 3 versus $\mathbf{c} 1$ (Plot1)? (Remember, the only difference between the two is the lower limit of integration.)
12. It appears that Plot1 and Plot2 differ by a constant, i.e., that $\mathbf{y} 3(x)-y^{2}(x)$ is the same for all values of $\mathbf{x}$. Find the value of this constant.
(You could calculate this in several different ways. If you get stuck, try expressing $y^{3}(x)-y^{2}(x)$ in terms of an integral.)
13. Explain why plots 1 and 2 have the same shape.

Look at the table of values for c3 and c4, and notice that the locations of the extrema are the same for both functions (as you would expect from an inspection of the scatterplots.)
14. Look at the locations of the local minima and the local maximum on the graph of $\mathbf{y} \mathbf{1}(\mathbf{x})$. At these values of $\mathbf{x}$, describe what appears to happen with the concavity of the graphs from Plot1 and Plot2.
15. Write an equation with a derivative on one side that shows how $\mathbf{y} \mathbf{1}(\mathbf{x})$ and $\mathbf{y} \mathbf{2 ( x )}$ are related.
16. Recall that the derivative of a function $f$ at $x=a$ can be defined by $f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$.

Approximate $\mathrm{y}^{2}$ (1) by evaluating the expression ( $\mathrm{y}^{2}(1.1)-\mathrm{y}^{2}(1)$ ) /0.1. $\qquad$ Your answer should be close to the value of $y \mathbf{1}(1)$. Why?
17. Approximate y3' (1) by evaluating the expression (y3(1.1)-y3(1))/0.1. Again, your answer should be close to the value of $y^{1}(1)$. Why?

## Teacher Notes, Answers, and Extensions

The version of the Fundamental Theorem covered by this activity says that if $f$ is a continuous function on the closed interval $[a, b]$, then the function $F(x)=\int_{a}^{x} f(t) d t$ is an antiderivative for $f$. Other ways to say this are $F^{\prime}(x)=f(x)$, or $\frac{d}{d x} \int_{a}^{x} f(t) d t=f(x)$. This is an important result in the history of the calculus, and one goal of this activity is to help students discover and appreciate its beauty. Students need to remain focused while carrying out the activity and think about each step.

## Class Management

This activity takes at least one full class period (not counting the extension) to complete ( 45 to 50 minutes). If one class period is insufficient, students could complete the activity as a homework assignment. However, they should have at least the lists created and the first six questions done in class. Teachers should work through the activity once before doing it with students. Depending on the calculator skills of your students, they could work independently, in groups, or with the teacher guiding the activity on the overhead display.

## Calculus Prerequisites

This activity can be done any time after students have learned about a function defined by a definite integral, like $F(x)=\int_{a}^{x} f(t) d t$. Students should know such functions will be positive when the integrand is positive and $x>a$, and the functions will be negative when the integrand is a negative function and $x>a$. Although not essential, it is helpful if students know the addition property:

$$
\int_{a}^{b} f(t) d t+\int_{b}^{c} f(t) d t=\int_{a}^{c} f(t) d t .
$$

## Calculator Prerequisites

This activity makes extensive use of calculator functionality. Students must be able to define and graph functions and scatterplots, set up a viewing window, and find the zeroes of a function on the graph screen. The activity introduces the calculator's integral function and shows how to create a list using the seq command.

## Preliminary Comments

This exploration gives students hands-on experience with functions of the form $F(x)=\int_{a}^{x} f(t) d t$ and emphasizes the connection between the function, $F$, and the integrand, $f$. That connection is at the heart of the Fundamental Theorem.

The hardest calculator work occurs before any questions are asked. If students get confused about what c3 represents, it may be helpful to have them evaluate $\int\left(y^{1}(t), t, 0,0.1\right)$, $\mathrm{y} 2(0.1)$, and intOouts [2] on the HOME screen. All three of those expressions are equal. The activity could be done using the calculator's built-in TABLE feature and by graphing y $\mathbf{y}^{2}$ and y3 directly, instead of as scatterplots. However, it takes a long time to redraw the graphs or redisplay the tables for functions defined by integrals. Since students are going back and forth between the graph and the table several times in the activity, the values of the integral are stashed away in lists. This process has a side benefit. It allows you to do calculations with those values. In particular, you can approximate the derivative of $\mathbf{y} 2$ by forming difference quotients. $A$ scatterplot of those approximations can then be compared with the graph of $\mathbf{y} 1$. This extension appears after the answers and supports the conclusion students make at step 15.

## Answers and Additional Comments

In the first six questions, students should begin to see that the function, $\mathbf{y} \mathbf{2}$, is increasing wherever the integrand is positive and decreasing wherever the integrand is negative. This connection should resonate with students' experience with increasing/decreasing behavior of a function and the sign of its derivative.

1. The first value in c 3 represents $\int_{0}^{0} \mathrm{y} \mathbf{1}(t) d t$ and so is zero. There is no area from 0 to 0 .
2. The second value in $\mathbf{c} 3$ is positive since the integrand is positive for all $t$ in the interval [0, 0.1].
3. The third value in c3 is greater than the second since the integrand is positive for all $t$ in the interval [0.1,0.2].
4. 1.8
5. $\int_{0}^{1.9} \cos \left(\frac{t^{2}}{2}\right) d t<\int_{0}^{1.8} \cos \left(\frac{t^{2}}{2}\right) d t$ because the integrand is negative for all $t$ in the interval [1.8,1.9]. Thus, the function is decreasing.
6. 3.1

In questions 7 through 10, students should recognize that the function y 2 has a maximum where the integrand changes sign from positive to negative and a minimum where the integrand changes sign from negative to positive. Again, this connection is meant to recall students' experience with the locations of maximum and minimum points and their relationship with the sign of the derivative. Ask questions like, "Have you ever seen a situation in which one 'thing' reaches a maximum where another 'thing' changes sign from positive to negative? What are the two 'things'?" Students make the connection explicit in question 15.
7. (a) 1.77245 and (b) 3.06998
8. At the first $x$-intercept, 1.77245 , the integrand changes sign from positive to negative. Thus the function $\mathrm{y}^{2}$ has a local maximum there. We stop accumulating positive values of y 1 and start accumulating negative values at that point. Similarly, at $x=3.06998$, the integrand changes sign from negative to positive. Thus we stop accumulating negative values and start accumulating positive values at that point, producing a local minimum on the graph of $\mathbf{y} 2$.
9. 3.963327
10. The function $\mathbf{y} 2$ has a maximum there. Looking at the table of values for $\mathbf{c} 1, \mathrm{c} 2$, and $\mathbf{c 3}$, this occurs near $\mathbf{x}=4$.
11. Answers will vary. Focusing on $\mathbf{x}=0, \mathbf{y}^{2}(0)=\int_{0}^{0} \cos \left(\frac{t^{2}}{2}\right) d t$ is zero, since the upper and lower limits of integration are equal. However,

$$
\mathrm{y} 3(0)=\int_{1}^{0} \cos \left(\frac{t^{2}}{2}\right) d t=-\int_{0}^{1} \cos \left(\frac{t^{2}}{2}\right) d t<0
$$

since the integral is positive. This puts the graph of $\mathbf{c 4}$ below the graph of $\mathbf{c 3}$. Also see the answer to question 12.
12. The difference is a constant. Just calculate $\mathrm{y} 3(0)-\mathrm{y}^{2}(0)=-0.975288$. Students may notice that

$$
\mathrm{y} 3(\mathbf{x})-\mathrm{y} 2(\mathbf{x})=\int_{1}^{x} \cos \left(\frac{t^{2}}{2}\right) d t-\int_{0}^{x} \cos \left(\frac{t^{2}}{2}\right) d t=-\int_{0}^{1} \cos \left(\frac{t^{2}}{2}\right) d t
$$

and simply evaluate the latter integral, or note that it is the negative of int1outs [1].
The punch line is at hand in question 13! The Fundamental Theorem says that the two functions, $y^{2}$ and $y^{3}$, have the same derivative, $y^{1}$. The derivative of a function tells you the slope on the graph of the function, and graphs that have the same slope everywhere have the same shape. In fact, Plot1 and Plot2 are members of the family of antiderivatives of $\mathbf{y} 1$. Two antiderivatives of a function always differ by a constant.
13. Plot1 and Plot2 have the same shape since each accumulates the same amount of area from the graph of $\mathbf{y} 1$ as $x$ increases. So each increases at the same rate where the integrand is positive, and each decreases at the same rate where the integrand is negative.

If you have addressed the connection between points of inflection and extrema of the first derivative (in addition to the more customary definition as points where the second derivative changes sign), then question 14 should give students yet another indicator that $\mathbf{y} 1$ is indeed the derivative of $\mathbf{y} \mathbf{2}$.
14. The graphs of Plot1 and Plot2 both appear to have a change of concavity at the points where the graph of $\mathbf{y} 1$ has a local maximum or a local minimum. The points of inflection occur at $x=2.506629, x=3.544908$, and $x=4.341608$. These are the extrema on the graph of y 1 .
15. $\frac{d}{d \mathrm{x}} \mathrm{y}^{2}(\mathrm{x})=\mathrm{y} \mathbf{1}(\mathrm{x})$.

Questions 16 and 17 lead students to approximate the derivative of $\mathrm{y}^{2}$ by forming difference quotients. The answers are not exactly the same as the output from $\mathbf{y} 1$. To get better agreement, you could have students form a difference quotient with a smaller step size as follows:

Evaluate $\frac{y^{2}(1.0001)-y^{2}(1)}{0.0001}$, and compare it to $y^{1}(1)$. They agree to four digits to the right of the decimal!
16. $\left(\mathrm{y}^{2}(1.1)-\mathrm{y}^{2}(1)\right) / 0.1=0.851261 . \mathrm{y}^{1}(1)=0.877583$.
17. $\left(y^{3}(1.1)-y^{3}(1)\right) / 0.1=0.851261 . y^{1}(1)=0.877583$.

Notice that the difference quotients are identical. y2and y3 are changing by exactly the same amount, an amount dictated by the magnitude of $\mathbf{y} 1$.

## Extension

After students have seen that $\frac{d}{d \mathbf{x}} \mathbf{y}^{\mathbf{2}(\mathbf{x})=\mathbf{y}^{\mathbf{1}}(\mathrm{x}) \text {, it is natural to gather more evidence in }}$ support of this conclusion. Because we have assembled data of inputs and outputs for $\boldsymbol{y}^{2}$, and we have a formula for $\mathbf{y} \mathbf{1}$, it is a relatively simple matter to form difference quotients for $\mathrm{y}^{2}$ and compare them to outputs from y 1 . This was done for a single value of $x$ in question 16 . Here's one way to do it for all the values of $\mathrm{y}^{2}$ that we've calculated.

Form the difference quotients using the calculator's delta-list command. That command generates a new list, with each entry formed by subtracting adjacent pairs of the argument. Applying delta-list to the output list intOouts, and then dividing that list by the differences in the input list, 0.1, gives a list of difference quotients. In the Data/Matrix Editor, highlight c5, press Enter, select $\Delta$ List ( from the CATALOG menu (under L), type int0outs, and divide by 0.1 . See the screen shots below.


Note that there will be one entry fewer in c5 than in c1, c2, c3, or c4. To compensate, you can enter one additional value at the end of c5 "by hand." To do this, you have to "unlock" the list c5 by highlighting c5 and pressing ENTER, CLEAR, and ENTER again. See the screen shots below.


Having unlocked c5, position the cursor in row 46 (r46c5, one cell past the last entry), press ENTER, and enter the expression ( $\mathrm{y}^{2}(4.6)-\mathrm{y}^{2}(4.5)$ )/0.1. This will even out the lists. Define Plot1 as c5 versus c1. See the screen shots below. In addition, deselect Plot2.


If you fail to do this correctly, you will probably get the error Dimension Mismatch, telling you the lengths of the lists you are trying to graph don't match. If this happens, fix the lists so they have the same length.

Finally, change the style of the graph of $\mathbf{y} \mathbf{1}$ from the $\mathbf{y}=$ editor by pressing F6:Style and selecting 6 : Path. That way, you'll be able to see the graph of $y 1$ running on top of the scatterplot. Deselect both y2 and y3. See the screen shots below.


Cool! What you just did was calculate 45 difference quotients for the function $y^{2}$. Each difference quotient value in the list c5 approximates the derivative of $\mathbf{y} 2$. You then graphed the
integrand on top of the scatterplot of those 45 difference quotients and saw the graphs coincide. This provides more evidence in support of the Fundamental Theorem of Calculus.

You could repeat this extension, using the outputs from the function defined in int1outs instead of int0outs. That is, form the difference quotients for int1outs this time, and store to c6. After unlocking c6, enter (y2 (4.6) -y2 (4.5) )/0.1 in r46c6. Set up a plot to graph c6 versus c1. The results should be identical, since int1outs and int0outs differ by a constant. In fact, you could calculate just what that constant is in two different ways: by subtracting the lists or by evaluating y2 (1) (which is, of course, an integral). See the screen shots below.


