2024

AP® Calculus BC

Scoring Guidelines

Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

	Model Solution	Scoring	
(a)	Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.		
	$C'(5) \approx \frac{C(7) - C(3)}{7 - 3} = \frac{69 - 85}{4} = -4$ degrees Celsius per minute	Estimate with supporting work	**1 point**
		Units	**1 point**

Scoring notes:

- To earn the first point a response must include a difference and a quotient as the supporting work.
- $\frac{-16}{7-3}$, $\frac{69-85}{7-3}$, or $\frac{69-85}{4}$ is sufficient to earn the first point.
- A response that presents only units without a numerical approximation for $C'(5)$ does not earn the second point.
- The second point is also earned for "degrees per minute" attached to a numerical value.

Total for part (a) 2 points

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

$\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7)$ $= 3\cdot 100 + 4\cdot 85 + 5\cdot 69 = 985$	Form of left Riemann sum	**1 point**
	Estimate	**1 point**
$\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval from $t = 0$ to $t = 12$.	Interpretation	**1 point**

Scoring notes:

- Read "$=$" as "$\approx$" for the first point.
- To earn the first point at least five of the six factors in the Riemann sum must be correct. If any of the six factors is incorrect, the response does not earn the second point.
- A response of $(3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7)$ earns the first point. Values must be pulled from the table to earn the second point.
- A response of $3\cdot 100 + 4\cdot 85 + 5\cdot 69$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A completely correct right Riemann sum (e.g., $3\cdot 85 + 4\cdot 69 + 5\cdot 55$) earns 1 of the first 2 points. An unsupported answer of 806 does not earn either of the first 2 points.
- Units will not affect scoring for the second point.
- To earn the third point the interpretation must include both "average temperature" and the time interval. The response need not include a reference to units. However, if incorrect units are given in the interpretation, the response does not earn the third point.

Total for part (b) 3 points

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

$C(20) = C(12) + \int_{12}^{20} C'(t)\,dt$	Integral	**1 point**
	Uses initial condition	**1 point**
$= 55 - 14.670812 = 40.329188$ The temperature of the coffee at time $t = 20$ is 40.329 degrees Celsius.	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral with integrand $C'(t)$. If the limits of integration are incorrect, the response does not earn the third point.
- A linkage error such as $C(20) = \int_{12}^{20} C'(t)\,dt = 55 - 14.670812$ or $\int_{12}^{20} C'(t)\,dt = -14.670812 = 40.329188$ earns the first 2 points but does not earn the third point.
- Missing differential (dt):
  - Unambiguous responses of $C(20) = C(12) + \int_{12}^{20} C'(t)$ or $C(20) = 55 + \int_{12}^{20} C'(t)$ earn the first 2 points and are eligible for the third point.
  - Ambiguous responses of $C(20) = \int_{12}^{20} C'(t) + C(12)$ or $C(20) = \int_{12}^{20} C'(t) + 55$ do not earn the first point, earn the second point, and earn the third point if the given numeric answer is correct. If there is no numeric answer given, these responses do not earn the third point.
- The second point is earned for adding $C(12)$ or 55 to a definite integral with a lower limit of 12, either symbolically or numerically.
- The third point is earned for an answer of $55 - 14.671$ or $-14.671 + 55$ with no additional simplification, provided there is some supporting work for these values.
- An answer of just 40.329 with no supporting work does not earn any points.

Total for part (c) 3 points

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Because $C''(t) > 0$ on the interval $12 < t < 20$, the rate of change in the temperature of the coffee, $C'(t)$, is increasing on this interval. That is, on the interval $12 < t < 20$, the temperature of the coffee is changing at an increasing rate.	Answer with reason	**1 point**

Scoring notes:

- This point is earned only for a correct answer with a correct reason that references the sign of the second derivative of C.
- A response that provides a reason based on the evaluation of $C''(t)$ at a single point does not earn this point.
- A response that uses ambiguous pronouns (such as "It is positive, so increasing") does not earn this point.
- A response does not need to reference the interval $12 < t < 20$ to earn the point.

Total for part (d) 1 point

Total for question 1 9 points

Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

	Model Solution	Scoring	
(a)	Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.		
	$\sqrt{(x'(2))^2 + (y'(2))^2}$	Setup for speed	**1 point**
	$= 12.3048506$ The speed of the particle at time $t = 2$ seconds is 12.305 (or 12.304) centimeters per second.	Answer	**1 point**

Scoring notes:

- The first point is earned for the expression $\sqrt{(x'(2))^2 + (y'(2))^2}$, $\sqrt{(x'(t))^2 + (y'(t))^2}$, or equivalent.
- A response that presents just the exact answer, $\sqrt{144 + \left(-2 + \sqrt{2^{1.2} + 20}\right)^2}$, earns both points.
- The second point is earned only for the answer 12.305 (or 12.304) regardless of whether the first point is earned.
- A response that includes a linkage error, such as $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{144 + \left(-2 + \sqrt{2^{1.2} + 20}\right)^2}$ or $\sqrt{(x'(t))^2 + (y'(t))^2} = 12.305$ (or 12.304), earns at most 1 of the 2 points.
- Missing or incorrect units will not affect scoring in this part.

Total for part (a) 2 points

(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$	Integral	**1 point**
$= 15.901715$ The total distance traveled by the particle over the time interval $0 \le t \le 2$ is 15.902 (or 15.901) centimeters.	Answer	**1 point**

Scoring notes:

- The first point is earned only for an integral of $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ (or the mathematical equivalent), with or without the differential.
  - Note: $\int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\, dt$ is not read as a parenthesis error.
- The second point is earned only for an answer of 15.902 (or 15.901), regardless of whether the first point is earned.
- Missing or incorrect units will not affect scoring in this part.

Total for part (b) 2 points

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

$y(0) = 6 + \int_2^0 y'(t)\, dt = 6 - 7.173613 = -1.173613$ The y-coordinate of the position of the particle at time $t = 0$ is -1.174 (or -1.173).	Definite integral	**1 point**
	Uses initial condition	**1 point**
	Answer	**1 point**

Scoring notes:

- An answer of -1.174 (or -1.173) with no supporting work does not earn any points.
- The first point is earned for either of the definite integrals $\int_2^0 y'(t)\, dt$ or $\int_0^2 y'(t)\, dt$.
- The second point is earned for any of:
  - $y(2) \pm \int_2^0 y'(t)\, dt$, $6 \pm \int_2^0 y'(t)\, dt$,
  - $y(2) \pm \int_0^2 y'(t)\, dt$, $6 \pm \int_0^2 y'(t)\, dt$,
  - $y(2) \pm 7.173613$, or 6 ± 7.173613.
- A response that attempts to evaluate $\int y'(t)\, dt$ does not earn the first or the third point.
  - Such a response can earn the second point by attempting to solve for the constant of integration by presenting an expression as an antiderivative for $y'(t)$, evaluating this expression at $t = 2$, and setting this expression equal to 6.

- A response that reverses the limits of integration, e.g., $y(0) = 6 + \int_0^2 y'(t)\,dt$ or $6 + 7.173613$, earns the second point but does not earn the third point.
- In order to earn the third point, a response must have earned at least 1 of the first 2 points.
- A response containing any linkage error can earn at most 2 of the 3 points. For example:
  - Equating two unequal quantities: $\int_2^0 y'(t)\,dt = -1.174$, $\int_2^0 y'(t)\,dt = 6 - 7.173613$, $6 + \int_0^2 y'(t)\,dt = 6 - 7.173613$, or $6 + \int_0^2 y'(t)\,dt = 6 + 7.173613 = -1.173613$
  - Equating an expression to a numerical value: $y(t) = 6 + \int_2^0 y'(t)\,dt = -1.174$
- Missing differentials (dt):
  - Unambiguous responses of $y(2) + \int_2^0 y'(t)$, $y(2) - \int_0^2 y'(t)$, $6 + \int_2^0 y'(t)$, or $6 - \int_0^2 y'(t)$ earn the first 2 points and would earn the third point for the correct numerical answer.
  - Unambiguous responses of $y(2) + \int_0^2 y'(t)$ or $6 + \int_0^2 y'(t)$ with reversed limits of integration and missing differential earn the first 2 points but cannot earn the third point.
  - Ambiguous responses of $\int_2^0 y'(t) + y(2)$, $-\int_0^2 y'(t) + y(2)$, $\int_2^0 y'(t) + 6$, or $-\int_0^2 y'(t) + 6$ earn the first point, do not earn the second point, but do earn the third point if a correct numeric answer is provided. If no numeric answer is given, none of these responses earn the third point.
  - Ambiguous responses of $\int_0^2 y'(t) + y(2)$ or $\int_0^2 y'(t) + 6$ with reversed limits of integration and no differential earn 1 out of 3 points.
- If a response provides work for both the x- and y-coordinates, the work for the x-coordinate will not affect scoring.
- However, a response that reports only a completely correct x-coordinate of the particle's position at time $t = 0$ with all supporting work, e.g., $x(0) = 3 + \int_2^0 x'(t)\,dt = -\frac{31}{3} = -10.333$, earns 2 out of 3 points.

Total for part (c) 3 points

(d) For $2 \le t \le 8,$ the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Because $y(t) > 0$ when $2 \le t \le 8,$ the particle will be moving toward the x-axis when $y'(t) < 0.$ This occurs when 5.222 (or $5.221) < t < 8.$	Considers sign of $y'(t)$	**1 point**
	Answer with reason	**1 point**

Scoring notes:

- The first point can be earned by stating $y'(t) = 0,$ $y'(t) < 0,$ $y'(t) > 0,$ or $t = 5.222$ (or 5.221).
  Note: $y'(t)$ may be written as $\frac{dy}{dt}.$
- The second point cannot be earned without the first.
- To earn the second point, a response must identify the correct interval (and no additional intervals in $[2, 8]$) and explicitly state the need for $y'(t) < 0.$ The interval can be open, closed, or half-open.

Total for part (d) 2 points

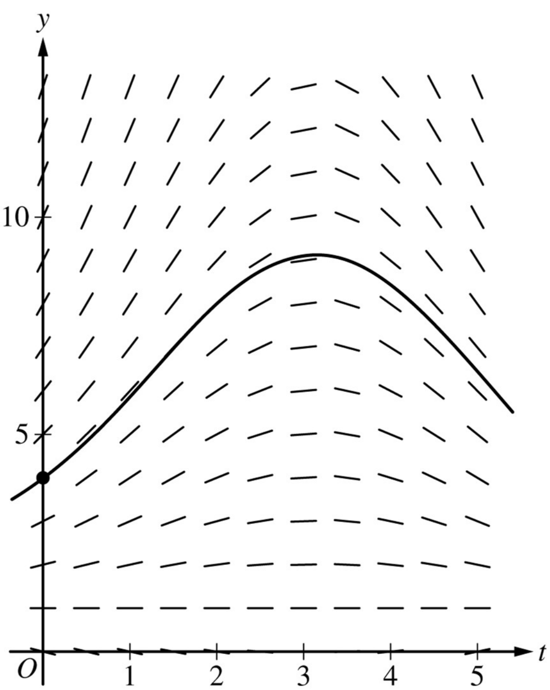
Total for question 2 9 points

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

	Model Solution	Scoring	
(a)	A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.		
		Solution curve	**1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 4)$, extend to at least $t = 4.5$, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.

Total for part (a) **1 point**

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Because $H(t) > 1$, then $\frac{dH}{dt} = 0$ implies $\cos\left(\frac{t}{2}\right) = 0$.	Considers sign of $\frac{dH}{dt}$	**1 point**
This implies that $t = \pi$ is a critical point.	Identifies $t = \pi$	**1 point**
For $0 < t < \pi$, $\frac{dH}{dt} > 0$ and for $\pi < t < 5$, $\frac{dH}{dt} < 0$. Therefore, $t = \pi$ is the location of a relative maximum value of H.	Answer with justification	**1 point**

Scoring notes:

- The first point is earned for considering $\frac{dH}{dt} = 0$, $\frac{dH}{dt} > 0$, $\frac{dH}{dt} < 0$, $\cos\left(\frac{t}{2}\right) = 0$, $\cos\left(\frac{t}{2}\right) > 0$, or $\cos\left(\frac{t}{2}\right) < 0$.
- The second point is earned for identifying $t = \pi$, with or without supporting work. A response may consider $H = 1$ or $t = 1$ as potential critical points without penalty.
- The third point cannot be earned without the first point. The third point is earned only for a correct justification and a correct answer of "relative maximum."
- The justification can be shown by determining the sign of $\frac{dH}{dt}$ (or $\cos\left(\frac{t}{2}\right)$) at a single value in $0 < t < \pi$ and at a single value in $\pi < t < 5$. It is not necessary to state that $\frac{dH}{dt}$ does not change sign on these intervals.
- The third point can also be earned by using the Second Derivative Test. For example:

$$\frac{d^2H}{dt^2} = \frac{1}{2}(H-1)\left(-\frac{1}{2}\sin\left(\frac{t}{2}\right)\right) + \cos\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dH}{dt}$$

$$\left.\frac{d^2H}{dt^2}\right|_{t=\pi} < 0$$

Therefore, $t = \pi$ is the location of a relative maximum value of H.

Total for part (b) 3 points

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

$\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$	Separation of variables	**1 point**
$\int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$ $\Rightarrow \ln\lvert H-1\rvert = \sin\left(\frac{t}{2}\right) + C$	One antiderivative	**1 point**
	Second antiderivative	**1 point**
$\ln\lvert 4-1\rvert = \sin\left(\frac{0}{2}\right) + C \Rightarrow C = \ln 3$ Because $H(0) = 4$, $H > 1$, so $\lvert H-1\rvert = H-1$. $\ln(H-1) = \sin\left(\frac{t}{2}\right) + \ln 3$	Constant of integration and uses initial condition	**1 point**
$H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$ $H(t) = 1 + 3e^{\sin(t/2)}$	Solves for H	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points.
- A response that presents $\int \frac{dH}{H-1} = \ln(H-1)$ without absolute value symbols earns that antiderivative point.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 0 for t and 4 for H.
- A response is eligible for the fifth point only if it has earned the first 4 points.
- A response earns the fifth point only for an answer of $H(t) = 1 + 3e^{\sin(t/2)}$ or a mathematically equivalent expression for $H(t)$ such as $H(t) = 1 + e^{\sin(t/2)+\ln 3}$.
- A response does not need to argue that $\lvert H-1\rvert = H-1$ in order to earn the fifth point.
- Special case: A response that presents an incorrect separation of variables of $\frac{1}{2}\cdot\frac{dH}{H-1} = \cos\left(\frac{t}{2}\right)dt$ does not earn the first point or the fifth point but is eligible for the 2 antiderivative points. If the response earns at least 1 of the 2 antiderivative points, then the response is eligible for the fourth point.

Total for part (c) 5 points

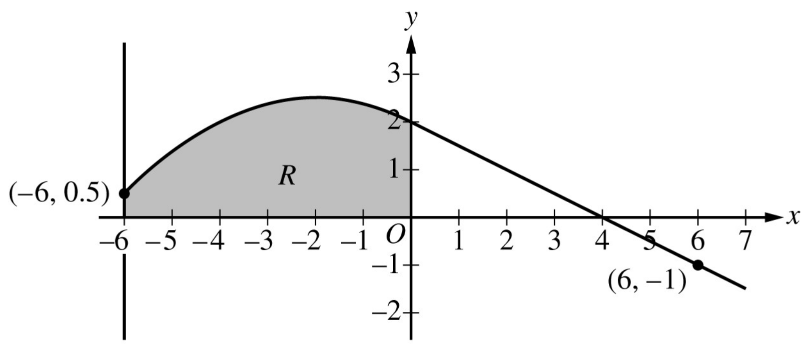
Total for question 3 9 points

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f

The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

Model Solution	Scoring	
(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.		
$g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$	$g(-6)$	**1 point**
$g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}\cdot 4\cdot 2 = 4$	$g(4)$	**1 point**
$g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}\cdot 4\cdot 2 - \frac{1}{2}\cdot 2\cdot 1 = 3$	$g(6)$	**1 point**

Scoring notes:

- Supporting work is not required for any of these values. However, any supporting work that is shown must be correct to earn the corresponding point.
- Special case: A response that explicitly presents $g(x) = \int_{-6}^{x} f(t)\,dt$ does not earn the first point it would have otherwise earned. The response is eligible for all subsequent points for correct answers, or for consistent answers with supporting work.
  - Note: $\int_{-6}^{-6} f(t)\,dt = 0$, $\int_{-6}^{4} f(t)\,dt = 16$, $\int_{-6}^{6} f(t)\,dt = 15$
- Labeled values may be presented in any order. Unlabeled values are read from left to right and from top to bottom as $g(-6)$, $g(4)$, and $g(6)$, respectively. A response that presents only 1 or 2 values must label them to earn any points.

Total for part (a) 3 points

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

$g'(x) = f(x)$	Fundamental Theorem of Calculus	**1 point**
$g'(x) = f(x) = 0 \Rightarrow x = 4$ Therefore, the graph of g has a critical point at $x = 4$.	Answer with reason	**1 point**

Scoring notes:

- The first point is earned for explicitly making the connection $g' = f$ in this part.
  - A response that writes $g'' = f'$ earns the first point but can only earn the second point by reasoning from $f = 0$.
- A response that does not earn the first point is eligible to earn the second point with an implied application of the FTC (e.g., "Because $g'(4) = 0$, $x = 4$ is a critical point").
- A response that reports any additional critical points in $0 < x < 6$ does not earn the second point.
  - Any presented critical point outside the interval $0 < x < 6$ will not affect scoring.

Total for part (b) 2 points

(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

$h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$	Uses Fundamental Theorem of Calculus	**1 point**
	$h(6)$ with supporting work	**1 point**
$h'(x) = f'(x)$, so $h'(6) = f'(6) = -\frac{1}{2}$.	$h'(6)$	**1 point**
$h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$.	$h''(6)$	**1 point**

Scoring notes:

- Labeled values may be presented in any order.
- Unlabeled values are read from left to right and from top to bottom as $h(6)$, $h'(6)$, and $h''(6)$, respectively. A response that presents only 1 or 2 values must label them in order to earn any points.
- A response of $h(6) = -1.5$ does not earn either of the first 2 points. A response of $h(6) = f(6) - f(-6)$ earns the first point but not yet the second point.
- A response of $h(6) = -1 - 0.5$ is the minimum work required to earn both of the first 2 points.
- To earn the third point a response must state either $h'(x) = f'(x)$ or $h'(6) = f'(6)$, and provide an answer of $-\frac{1}{2}$.
- The fourth point is earned for a response of $h''(6) = 0$, with or without supporting work.
- A response that has one or more linkage errors does not earn the first point it would have otherwise earned. For example, $h'(x) = f'(6) = -\frac{1}{2}$ does not earn the third point but is eligible for the fourth point even in the presence of another linkage error, such as $h''(x) = f''(6) = 0$.

Total for part (c) 4 points

Total for question 4 9 points

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

x	0	π	2π
$f'(x)$	5	6	0

The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

	Model Solution	Scoring	
(a)	For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.		
	$h'(x) = \sqrt{1 + (f'(x))^2}$	Fundamental Theorem of Calculus	**1 point**
	$h'(\pi) = \sqrt{1 + (f'(\pi))^2} = \sqrt{1 + 6^2} = \sqrt{37}$	Answer	**1 point**

Scoring notes:

- A response of $\sqrt{1 + (f'(\pi))^2}$ earns the first point.
- A response of $\sqrt{1 + 6^2}$ alone earns both points.
- A response such as $h'(x) = \sqrt{1 + (f'(x))^2} = \sqrt{37}$, that equates a variable expression to a numeric value, earns at most 1 of the 2 points.
- A response that equates $h'(x)$ or $h'(\pi)$ to a derivative of a constant, such as $h'(x) = \frac{d}{dx}\int_0^\pi \sqrt{1 + (f'(t))^2}\, dt$, earns at most 1 of the 2 points.

Total for part (a) — 2 points

(b) What information does $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

$\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the arc length of the graph of f on $[0, \pi]$.	Arc length of f	**1 point**
	Interval $[0, \pi]$	**1 point**

Scoring notes:

- A response of "arc length" or "length" earns the first point. Such a response does not need to reference f. However, if the response references a different function, the response does not earn the first point and is eligible to earn the second point.
- A response referring to distance explicitly connected to the graph or f (or equivalent) earns the first point. For example, a response of "distance along the curve" or "distance traveled by a particle moving along f" earns the first point and is eligible to earn the second point.
- A response referring to distance that is not explicitly connected to the graph of f does not earn the first point but is eligible to earn the second point. For example, a response of "distance" or "distance traveled" does not earn the first point but is eligible to earn the second point.
- To earn the second point a response must connect the interval $[0, \pi]$ to arc length, length, or distance.

Total for part (b) 2 points

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

$f(\pi) \approx f(0) + \pi f'(0) = 0 + 5\pi = 5\pi$ $f(2\pi) \approx f(\pi) + \pi f'(\pi)$	Euler's method	**1 point**
$\approx 5\pi + 6\pi = 11\pi$	Answer	**1 point**

Scoring notes:

- To earn the first point a response must demonstrate two Euler's steps, with use of the correct expression for $\frac{dy}{dx}$, and at most one error. If there is an error, the second point is not earned.
- In order to earn the first point, a response that presents a single error in computing the approximation of $f(\pi)$ must import the incorrect value in computing the approximation of $f(2\pi)$.
- The two Euler's steps may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot \pi$)
0	0	5π
π	5π	6π
2π	11π	

  - In the presence of a correct answer, a table does not need to be labeled in order to earn both points. In the presence of no answer or an incorrect answer, such a table must be correctly labeled in order to earn the first point.
- Both points are earned for $5\pi + 6\pi$.
- The response may report the final answer as $(2\pi, 11\pi)$.

Total for part (c) 2 points

(d) Find $\int (t+5)\cos\left(\frac{t}{4}\right)\,dt$. Show the work that leads to your answer.

$u = t + 5 \qquad dv = \cos\left(\frac{t}{4}\right)dt$ $du = dt \qquad v = 4\sin\left(\frac{t}{4}\right)$	u and dv	**1 point**
$\int (t+5)\cos\left(\frac{t}{4}\right)dt = 4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt$	$uv - \int v\,du$	**1 point**
$= 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$	Answer	**1 point**

Scoring notes:

- The first and second points are earned with an implied u and dv in the presence of $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt$ or a mathematically equivalent expression.
- The tabular method may be used to show integration by parts. In this case, the first point is earned by columns (labeled or unlabeled) that begin with $t+5$ and $\cos\left(\frac{t}{4}\right)$. The second point is earned for $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt$ or a mathematically equivalent expression.
- The third point is earned only for an expression mathematically equivalent to $4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$ (such as $4t\sin\left(\frac{t}{4}\right) + 20\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$) in the presence of correct supporting work.
- To earn the third point a response must have a final answer that includes a constant of integration.
- Alternate solution:

$$\int (t+5)\cos\left(\frac{t}{4}\right)dt = \int t\cos\left(\frac{t}{4}\right)dt + \int 5\cos\left(\frac{t}{4}\right)dt$$

$$u = t \qquad dv = \cos\left(\frac{t}{4}\right)dt$$

$$du = dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

$$\int t\cos\left(\frac{t}{4}\right)dt + \int 5\cos\left(\frac{t}{4}\right)dt = 4t\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)dt + \int 5\cos\left(\frac{t}{4}\right)dt$$

$$= 4t\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + 20\sin\left(\frac{t}{4}\right) + C$$

  - A response can earn the first and second points for correctly applying integration by parts to $\int t\cos\left(\frac{t}{4}\right)dt$. The tabular method may be used to show integration by parts. The third point is earned for the correct answer.

Total for part (d) 3 points

Total for question 5 9 points

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

Model Solution	Scoring	
At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.	Considers $\frac{(n+1)6^n}{n^2 6^n}$	**1 point**
Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \geq 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the comparison test.	Answer with reason	**1 point**

Scoring notes:

- To earn the first point using either the comparison or limit comparison test, a response must consider the term $\frac{(n+1)6^n}{n^2 6^n}$. This could be shown by considering the term $\frac{n+1}{n^2}$, either individually or as part of a sum.
- To earn the second point using the comparison test a response must demonstrate that the terms $\frac{n+1}{n^2}$ are larger than the terms in a divergent series.
  - "$\frac{n+1}{n^2} > \frac{1}{n}$, diverges" earns both points.
  - The response does not need to use the term "comparison test," but the response cannot declare use of an incorrect test.
- Alternate solution (limit comparison test):

  At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.

  Because $\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the limit comparison test.

  - To earn the second point using the limit comparison test, a response must correctly write the limit of the ratio of the terms in the given series to the terms of a divergent series and demonstrate that the limit of this ratio is 1.
  - The response does not need to use the term "limit comparison test," but the response cannot declare use of an incorrect test.

Total for part (a) 2 points

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

Solution	Scoring
$f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}\left(-\frac{1}{2}\right)^n$ is an alternating series with terms that decrease in magnitude to 0. By the alternating series error bound, $\sum_{n=1}^{3} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n = -\frac{125}{144}$ approximates $f(-3)$ with error of at most $\left\lvert \frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4 \right\rvert = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$. Thus, $\lvert f(-3) - S_3 \rvert < \frac{1}{50}$.	Uses fourth term **1 point** Verification **1 point**

Scoring notes:

- The first point is earned for correctly using $x = -3$ in the fourth term. (Listing the fourth term as part of a polynomial is not sufficient.) Using $x = -3$ in any term of degree five or higher does not earn this point.
- The expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ earns the first point, but just $\frac{5}{256}$ does not earn the first point.
- A response including the expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ that is subsequently simplified incorrectly earns the first point but not the second.
- To earn the second point the response must state that the series for $f(-3)$ is alternating or that the alternating series error bound is being used.
    - A response of just "Error $\le \frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4 < \frac{1}{50}$" (or any equivalent mathematical expression) earns both points, provided it is accompanied by an indication that the series is alternating.
- A response that declares the error is equal to $\frac{5}{256}$ (or any equivalent form of this value) does not earn the second point.

Total for part (b) 2 points

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

The general term of the Maclaurin series for f' is $\frac{(n+1)nx^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n \cdot 6^n}$.	General term	**1 point**
Because the radius of convergence of the Maclaurin series for f is 6, the radius of convergence of the Maclaurin series for f' is also 6.	Radius	**1 point**

Scoring notes:

- A response of $\frac{(n+1)nx^{n-1}}{n^2 6^n}$ earns the first point. Any expression mathematically equivalent to this also earns the first point.
- The response need not simplify $\frac{(n+1)nx^{n-1}}{n^2 6^n}$, but any presented simplification must be correct in order to earn the first point.
- The second point is earned only for a supported answer of 6. The second point can be earned without the first.
- Alternate solution for second point (ratio test):

$$\lim_{n\to\infty}\left|\frac{\frac{(n+2)x^n}{(n+1)6^{n+1}}}{\frac{(n+1)x^{n-1}}{n\cdot 6^n}}\right| = \lim_{n\to\infty}\left|\frac{n(n+2)}{(n+1)^2}\cdot\frac{x}{6}\right| = \frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

Therefore, the radius of convergence of the Maclaurin series for f' is 6.

- Alternate solution for second point (root test):

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)|x|^{n-1}}{n\cdot 6^n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1/n}\cdot|x|^{-1/n}\cdot\frac{|x|}{6} = 1\cdot 1\cdot\frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

Therefore, the radius of convergence of the Maclaurin series for f' is 6.

Total for part (c) 2 points

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \frac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}} \right\rvert = \left\lvert \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}} \right\rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \left\lvert \frac{(n+2)n^2}{(n+1)^3} \right\rvert \cdot \left\lvert \frac{x^2}{3} \right\rvert = \left\lvert \frac{x^2}{3} \right\rvert$	Limit	**1 point**
$\left\lvert \frac{x^2}{3} \right\rvert < 1 \Rightarrow x^2 < 3 \Rightarrow \lvert x \rvert < \sqrt{3}$ The radius of convergence of g is $\sqrt{3}$.	Radius of convergence	**1 point**

Scoring notes:

- The first point is earned by presenting a correct ratio with or without absolute values. Once earned, this point cannot be lost. Any errors in simplification or evaluation of the limit will not earn the second point.
- The first point is earned for ratios mathematically equivalent to any of the following:

$$\frac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}}, \quad \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}, \quad \frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}}, \text{ or } \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n-1)^2 3^{n-1}}{nx^{2n-2}}.$$

- The first point is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}, \quad \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n+1)^2 3^{n+1}}{(n+2)x^{2n+2}}, \quad \frac{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}}, \text{ or } \frac{nx^{2n-2}}{(n-1)^2 3^{n-1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}.$$

  Responses including any of these reciprocal ratios can earn the second point for using limit notation to correctly find a limit of the absolute value of their ratio to be $\left\lvert \frac{3}{x^2} \right\rvert$. Such responses earn the third point only for a final answer of $\sqrt{3}$ with a valid explanation for reporting the reciprocal of $\frac{1}{\sqrt{3}}$.

- To earn the second point a response must use the ratio and correctly evaluate the limit of the ratio, using correct limit notation.
- The third point is earned only for an answer of $\sqrt{3}$ with supporting work.

Total for part (d) 3 points

Total for question 6 9 points