Chief Reader Report on Student Responses:
2024 AP® Calculus AB/BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC): 1,572

Calculus AB

- Number of Students Scored: 278,657
- Score Distribution

Exam Score	N	%At
5	59,569	21.4
4	77,458	27.8
3	42,533	15.3
2	63,178	22.7
1	35,919	12.9

- Global Mean: 3.22

Calculus BC

- Number of Students Scored: 148,191
- Score Distribution

Exam Score	N	%At
5	70,723	47.7
4	31,217	21.1
3	17,880	12.1
2	20,668	13.9
1	7,703	5.2

- Global Mean: 3.92

Calculus BC Calculus AB Subscore

- Number of Students Scored: 148,191
- Score Distribution

Exam Score	N	%At
5	74,262	50.1
4	43,966	29.7
3	12,508	8.4
2	13,598	9.2
1	3,857	2.6

- Global Mean: 4.16

* The number of students with Calculus AB subscores may differ slightly from the number of students who took the AP Calculus BC Exam due to exam administration incidents.

The following comments on the 2024 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Julie Clark of Hollins University. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1

Topic: Modeling – Rate of Change, Riemann Sum, Average Value
Max Score: 9
Mean Score AB1: 4.71
Mean Score BC1: 6.45

What were the responses to this question expected to demonstrate?

In this question students were given a table of times t in minutes, $0 \le t \le 12$, and values of a decreasing differentiable function $C(t)$ that models the temperature, in degrees Celsius, of coffee in a cup.

In part (a) students were asked to approximate the value of $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$ and to include correct units with their answer. A correct response will use values from the given table to calculate $C'(5) \approx \dfrac{C(7) - C(3)}{7 - 3} = \dfrac{69 - 85}{4} = -4$ degrees Celsius per minute.

In part (b) students were asked to approximate the value of $\int_0^{12} C(t)\, dt$ using a left Riemann sum with the three subintervals indicated by the values in the given table. Then students were asked to interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\, dt$ in the context of the problem. A correct response would present the left Riemann Sum setup and the approximation (e.g., $(3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7) = 985$). A correct response would also indicate that $\frac{1}{12}\int_0^{12} C(t)\, dt$ represents the average temperature of the coffee, in degrees Celsius, over the time interval from $t = 0$ to $t = 12$.

In part (c) $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$ was introduced as a function that models the rate of change of the coffee's temperature, in degrees Celsius per minute, over the time interval $12 \le t \le 20$. Students were asked to find the temperature of the coffee at time $t = 20$. A correct response would provide the setup $C(20) = C(12) + \int_{12}^{20} C'(t)\, dt$, then use a calculator to add the value $C(12) = 55$ to the value of integral and report a temperature of 40.329 degrees Celsius.

In part (d) students were given $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$, the derivative of the model introduced in part (c), and asked to determine whether the temperature of the coffee was changing at a decreasing rate or at an increasing rate for $12 < t < 20$. A correct response would observe that the given function, $C''(t)$, is positive on the interval $12 < t < 20$, and therefore the rate of change of the temperature of the coffee, $C'(t)$, is increasing on this interval.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

<u>Part (a)</u>

- Almost all responses were able to earn the first of the two available points in this part by finding the average rate of change and providing the supporting work of a difference quotient.
- Most responses completely simplified their approximation of $C'(5)$ and reached a correct value of –4.
- Overall, responses were more successful providing a correct approximation of $C'(5)$ than with providing correct units.
- Most responses recognized that the units would be a ratio, but incorrect responses often made errors in the numerator, the denominator, or both.

<u>Part (b)</u>

- A majority of the responses successfully provided a left Riemann sum approximation of $\int_0^{12} C(t)\,dt$.
- Although simplification of the approximation was not necessary, a majority of the responses did simplify and simplified correctly.
- In general, responses had difficulty providing a correct interpretation, although quite a few responses did provide a correct interpretation that included the necessary key words/phrases of "average temperature" and "interval from $t = 0$ to $t = 12$."

Part (c)

- Nearly all responses presented a setup including a definite integral with the correct integrand, $C'(t)$.
- A majority of the responses provided correct limits of integration as part of their integral.
- Most of the responses incorporated the initial condition in a valid manner.
- Some responses presented communication errors such as $\int_{12}^{20} C'(t)\,dt = 55 - 14.6708,$ and a very small number of responses reported an incorrect value of $\int_{12}^{20} C'(t)\,dt$ or rounded the value of $\int_{12}^{20} C'(t)\,dt$ prematurely.

Part (d)

- Some responses did correctly appeal to the positive sign of the given function, $C''(t)$, on the interval $12 < t < 20$ as the reason the temperature of the coffee was changing at an increasing rate.
- A significant number of responses incorrectly reasoned that the temperature of the coffee was changing at an increasing rate because of the opposite signs of $C''(t)$ and $C'(t)$ on this interval.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses correctly found a difference quotient of $\frac{69-85}{7-3}$ but went on to calculate an estimated temperature of $C(5) = 77.$ Then they boxed or circled the second value as a final answer. Such responses did not earn the first point in this part of the question.	• $C'(5) \approx \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4}$
• A few responses provided an average rate of change over an interval other than $3 \le t \le 7$, e.g., $\frac{C(12)-C(3)}{12-3} = \frac{55-85}{9}$ or $\frac{C(7)-C(0)}{7-0} = \frac{69-100}{7}$.	• $C'(5) \approx \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4}$
• Some responses reversed the order of subtraction in their difference quotient, providing a positive average rate of change: $\frac{85-69}{7-3} = 4.$	• $C'(5) \approx \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4} = -4$

• Many responses presented incorrect units such as $\frac{\text{degrees Celsius}}{\text{minutes}^2}$ or degrees/second.	• $-4\ \frac{\text{degrees Celsius}}{\text{minute}},\ -4\ \frac{\text{degrees}}{\text{minute}},$ or $-4^\circ/\text{min}.$
• In part (b) some responses did not provide a sufficient setup for a left Riemann sum, e.g., $300 + 340 + 345.$ A sufficient setup would show the sum of three appropriate products. • A small number of responses provided a right Riemann sum, e.g., $3 \cdot 85 + 4 \cdot 69 + 5 \cdot 55.$ • Some responses incorrectly used a common width of 3 for all subintervals in a left Riemann sum, e.g., $3(100 + 85 + 69).$ • Some responses used incorrect notation such as $\int_0^{12} (3 \cdot 100 + 4 \cdot 85 + 5 \cdot 69)\, dt.$	• $\int_0^{12} C(t)\, dt \approx (3-0) \cdot C(0) + (7-3) \cdot C(3) + (12-7) \cdot C(7)$ • $\int_0^{12} C(t)\, dt \approx 3 \cdot 100 + 4 \cdot 85 + 5 \cdot 69$ • $3 \cdot 100 + 4 \cdot 85 + 5 \cdot 69$ • $\int_0^{12} C(t)\, dt \approx 3 \cdot 100 + 4 \cdot 85 + 5 \cdot 69$
• In part (c) a few responses failed to provide a definite integral with integrand $C'(t)$ as part of a calculation setup. • Some responses used imprecise mathematical language by equating two unequal expressions, e.g., $\int_{12}^{20} C'(t)\, dt = 55 - 14.6708.$ • Some responses tried to integrate the function $C'(t)$ from $t = 0$ to $t = 12$ and/or to $t = 20$ but did not realize that both $\int_0^{12} C'(t)\, dt$ and $\int_0^{20} C'(t)\, dt$ are undefined and that $C'(t)$ only modeled the temperature's rate of change on the interval $12 \le t \le 20.$	• $C(20) = C(12) + \int_{12}^{20} C'(t)\, dt$ • $C(20) = 55 + \int_{12}^{20} C'(t)\, dt = 55 - 14.6708$ • $C(20) = 55 + \int_{12}^{20} C'(t)\, dt = 55 - 14.6708$
• In part (d) a large number of responses concluded the temperature of the coffee was changing at a decreasing rate because $C'(t) < 0$ and $C''(t) > 0$ on the interval $12 < t < 20.$ • Some responses reported that the temperature of the coffee was decreasing on this interval because $C'(t) < 0,$ which was true, but did not answer what the question asked.	• Because $C''(t) > 0$ on the interval $12 < t < 20,$ the temperature of the coffee is changing at an increasing rate during this time period. • Although $C'(t) < 0$ and therefore the temperature of the coffee is decreasing, $C''(t) > 0$ on the interval $12 < t < 20,$ so the temperature of the coffee is changing at an increasing rate during this time period.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Student responses demonstrated widespread skill using calculators to find the values of definite integrals and derivatives at a point. When asked to explain the meaning of these values, however, student responses frequently uncover gaps in understanding. An understanding of how derivatives and integrals create meaning in context is an important skill for students to develop. Being able to interpret meaning from existing units of measure and provide appropriate units after integrating or differentiating allows students to apply calculus to real-world problems and understand why calculus is an important part of mathematical study.
- Reviewing past years' Scoring Guidelines and Chief Reader Reports are the best ways to see exactly the type of terminology and specific language that is likely to earn points for questions asking for units, interpretations, and reasoning. Exposing students to the correct way to perform these tasks as soon as the concepts are taught is a great way to form good habits in writing these verbal explanations.
- Offering students practice applying the Fundamental Theorem of Calculus to find net change can foster student confidence in answering questions in applied contexts like part (c) of this problem.
- An effective teaching strategy to assist students in finding correct units is to have them circle or underline the units shown in a table or as labels along the axes in a graph. Depending upon whether the question asks for a derivative or an antiderivative, students can then either divide or multiply those units, respectively.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

AP® Calculus free response questions typically include a few points that should be relatively easily earned by students who have basic levels of understanding of the content and mastery of the skills of the course. Most students were able to earn at least one point on this question. It is important to build a strong foundation throughout the year and to practice identifying and earning those "entry" points on the exam.

- AP Student Practice is a resource that can meet the needs of students of all levels of understanding and mastery. Students can access this resource whether a teacher assigns it or not. Each unit has AP Student Practice questions pitched to three levels. We hope to see students build strong foundations in the basics and to advance to more challenging levels over time. These resources can be found on the relevant unit pages on AP Classroom.
    - AP Student Practice for Unit 2 Level 1 can help students with emerging understanding make sense of what is being asked in part (a) of this question and to build on that understanding with practice.
    - AP Student Practice for Unit 8 Level 1 can help students develop their interpretation skills, among others.
- Practice reading and working past AP® Calculus free response questions can help students to identify what is being asked in each part, relevant information they are given in the stem, and the types of responses they are expected to provide. This thoughtful pre-read will almost assure that students who have built a strong foundation across the year will find and earn points at all but the highest levels of difficulty.

Question AB2

Topic: Particle Motion – Acceleration, Position, Distance
Max Score: 9
Mean Score: 4.24

What were the responses to this question expected to demonstrate?

In this question students were told that a particle is moving along the x-axis with a velocity for $t \geq 0$ given by $v(t) = \ln\left(t^2 - 4t + 5\right) - 0.2t.$

In part (a) students were asked to find the one time, t_R, in the interval $0 < t < 2$ when the particle is at rest (not moving). Students were also asked to determine whether the particle is moving to the right or to the left for $0 < t < t_R$. A correct response would find the value of t_R by using a calculator to solve the equation $v(t) = 0$ for t. The response would continue by using the calculator to determine that the particle's velocity is positive for $0 < t < t_R$, which means the particle is moving to the right during this time interval.

In part (b) students were asked to find the acceleration of the particle at time $t = 1.5$ and to determine whether the speed of the particle is increasing or decreasing at this time. A correct response would indicate that the particle's acceleration is the derivative of the particle's velocity, $a(t) = v'(t)$, and would use a calculator to find the value $v'(1.5) = -1$. A correct response would continue by using the calculator to determine that $v(1.5) < 0$, so the particle's velocity and acceleration have the same sign when $t = 1.5$. Therefore, the speed of the particle is increasing then.

In part (c) students are asked to find the position of the particle at time $t = 4$, given that the position of the particle at time $t = 1$ is $x(1) = -3$. A correct response would provide the setup $x(4) = x(1) + \int_1^4 v(t)\,dt$ and use a calculator to find the value $-3 + 0.197 = -2.803$.

In part (d) students are asked to find the total distance traveled by the particle over the time interval $1 \leq t \leq 4$. A correct response would provide the setup $\int_1^4 |v(t)|\,dt$ and then use a calculator to find a total distance of 0.958.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

Part (a)

- Overall, the responses indicated a very good understanding that finding when a particle is at rest requires determining when the particle's velocity is zero.
- Additionally, most responses indicated that the particle was moving to the right because its velocity was positive on the given interval.
- Some responses reported the correct value of t_R but did not provide any explanation of how this value was found.

Part (b)

- Most responses used a calculator to find the correct value of $a(1.5)$, and many provided the setup for this calculation by reporting some version of $a(t) = v'(t)$.
- Some responses found the value of $a(1.5)$ without using a calculator by reporting

  $$v'(1.5) = \frac{2(1.5) - 4}{1.5^2 - 4(1.5) + 5} - 0.2.$$

- There were a reasonable number of responses that compared the signs of the particle's velocity and acceleration at time $t = 1.5$ and correctly concluded the particle's speed was increasing. However, quite a few responses considered only the sign of $a(1.5)$ to draw a conclusion about whether the particle's speed was increasing or decreasing.
- Some responses presented poor mathematical communication in the form of false statements such as $a(t) = v'(t) = -1.$

Part (c)

- A large number of responses correctly calculated the particle's position as $x(1) + \int_1^4 v(t)\, dt = -3 + 0.197$ and provided the correct setup, although many responses had an incorrect or missing differential dt.
- Some responses provided the setup $x(4) = x(0) + \int_0^4 v(t)\, dt$ and then used the given information that $x(1) = -3$ to find the value of $x(0)$ after using the calculator to find the value of $\int_0^1 v(t)\, dt.$ This is a valid approach, but is definitely more time consuming, and may represent a lack of understanding that the particle's position can be found using a definite integral with a lower limit of integration other than zero.
- Some responses approached this problem by using a computer algebra system (CAS) enabled calculator to find a closed form expression for $x(t) + C$ and then attempted to use $x(1) = -3$ to find the value of C and compute $x(4).$ This approach was usually not successful.

Part (d)

- Most responses presented a correct setup for the total distance traveled by the particle, and most also went on to correctly use a calculator to compute that distance.
- Some responses did not seem to know how to apply absolute values appropriately, incorrectly claiming the total distance traveled was $\left|\int_1^4 v(t)\right| dt.$

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses erroneously dropped the $-0.2t$ term from the velocity function. These students might have noticed their mistake if they had recognized that the value of t_R they found did not lie within the given interval $0 < t < 2.$ However, most did not. • Many responses did not explicitly communicate that the value t_R is found by determining when $v(t) = 0.$ • Some responses communicated poorly with statements such as "the velocity is moving right."	• The particle is at rest when its velocity is zero: $v(t) = 0 \Rightarrow \ln\left(t^2 - 4t + 5\right) - 0.2t = 0 \Rightarrow t = 1.4256.$ • The particle is at rest when its velocity is zero: $v(t_R) = 0 \Rightarrow t_R = 1.4256.$ • Because the particle's velocity is positive on the interval $0 < t < t_R,$ the particle is moving to the right.
• In part (b) some responses considered only the sign of $a(1.5)$ and concluded the particle's speed was decreasing when $t = 1.5.$ • Some responses provided (incorrect) global arguments about the signs of the particle's velocity and	• Because $a(1.5) < 0$ and $v(1.5) < 0,$ the speed of the particle is increasing at time $t = 1.5.$

acceleration, i.e., $a(t) < 0$ and $v(t) < 0$, without reference to the time $t = 1.5$. • Many responses used poor notation such as $\frac{dy}{dx}(v(t))$ or $v(t)\frac{dy}{dx}$ to represent the derivative of the velocity with respect to time. • Quite a few responses included imprecise mathematical language, such as $a(t) = v'(t) = -1$.	• Because $a(t) < 0$ and $v(t) < 0$ at time $t = 1.5$, the speed of the particle is increasing at time $t = 1.5$. • $a(t) = v'(t)$ or $a(t) = \frac{d}{dt}[v(t)]$ • $a(t) = v'(t)$; $a(1.5) = -1$ or $a(1.5) = v'(1.5) = -1$
• In part (c) many responses tried to find the position of the particle using $\int_0^4 v(t)\,dt$, despite having been given the initial condition $x(1) = -3$. • Many responses, such as $-3 + \int_1^4 v(t)\,dx$, $-3 + \int_1^4 v(t)$, or $\int_1^4 v(t) - 3$, had incorrect or missing differentials.	• $x(4) = x(1) + \int_1^4 v(t)\,dt$ • $x(4) = x(1) + \int_1^4 v(t)\,dt$, $x(4) = x(1) + \int_1^4 v(x)\,dx$, or $x(4) = \int_1^4 v(t)\,dt + x(1)$.
• In part (d) some responses provided a setup for the net distance traveled, $\int_1^4 v(t)\,dt$. This setup was sometimes seen with a correctly calculated value for the total distance traveled and sometimes seen with an incorrect value of 0.197. • Some responses mishandled the absolute values, e.g., $\left\lvert \int_1^4 v(t)\,dt \right\rvert = 0.197$.	• Total distance traveled $= \int_1^4 \lvert v(t) \rvert\,dt = 0.958$ • Total distance traveled $= \int_1^4 \lvert v(t) \rvert\,dt = 0.958$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers could provide students with multiple opportunities to connect the concepts of acceleration, displacement, total distance traveled, and position—pointing out the similarities between these concepts and how various derivatives and definite integrals involving velocity and speed correspond to these values.
- Students should be encouraged to always specify the interval or value(s) that are discussed in the question.
- Teachers could model precise and accurate communication and require such communication in all student work:
    - Students need to avoid general statements such as $a(t) = -1$ when they are being asked about the acceleration at a point. Generally, the statement $a(t) = -1$ means that the acceleration is constant; it must be qualified by adding "at $t = 1.5$." Encourage students to be careful about writing "$a(t) = \ldots$" when they mean "$a(1.5) = \ldots$."
    - Require students to provide accurate language. For example, the statement "the particle is increasing" is not correct, whereas "the speed of the particle is increasing" is correct.
    - Encourage students to use clear, unambiguous language. For example, "it" without a clear, single antecedent doesn't always qualify as proper communication as to what may be increasing, decreasing, positive, etc.

- Teachers might remind students that a CAS Calculator is NOT required and that using the CAS features may make the problem more complicated than intended.
- Some students would benefit from a more sophisticated understanding of numerical integration on a calculator. When integrating to find total distance traveled without a calculator, the student might need to break the interval into multiple pieces and integrate each piece individually. When using a calculator, a single integral with an integrand of $|v(t)|$ is sufficient and more efficient.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

This question assessed content and skills related to solving linear particle motion problems and students' mathematical communication skills. AP Classroom offers resources to develop understanding of specific content and skills, while the AP Calculus AB and BC Course and Exam Description (CED) provides strategies to help students develop good communication habits and to refine communication skills.

- Topic 8.2 on AP Classroom contains a link to a lesson and a student handout: *Analyzing Problems Involving Definite Integrals and Motion.*
- Page 212 of the CED describes the use of "sentence starters" as a way to help students to start on the right foot as they begin to work on mathematical communication skills: "Students respond to a prompt by filling in the missing parts of a given sentence template." Past AP exam free response questions that ask students to "give a reason" or "explain your reasoning" may be repurposed as sentence starters. For example, a sentence starter for part (a) of this question might be: "The particle is at rest at time t_R =_____, because _______. The particle is moving to the _____ on the interval $0 < t < t_R$ because _______. Students can learn to create these for themselves using language from the stem of the question.
- The Instructional Approaches section of the CED provides instructional strategies to help develop the Mathematical Practices (pages 214-220). Suggested strategies for developing Practice 4 (Communication and Notation) are found on page 220. Error analysis is recommended to develop skills important in this question: Skill 4.A (Use precise mathematical language) and skill 4.C (Use appropriate mathematical symbols and notation). Error analysis is described on page 206 of the CED: "Students analyze an existing solution to determine whether (or where) errors have occurred."

Question AB3/BC3

Topic: Modeling with Differential Equation – Separation of Variables
Max Score: 9
Mean Score AB3: 3.55
Mean Score BC3: 5.57

What were the responses to this question expected to demonstrate?

In this question students were told that the depth of sea water, in feet, could be modeled by the function H which satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where t is measured in hours after noon. Furthermore, $H(0) = 4$, and so at noon the depth of the seawater is 4 feet.

In part (a) students were given a slope field for the differential equation and asked to sketch the solution curve through the point $(0, 4)$. A correct response will draw a curve that passes through the point $(0, 4)$, follows the indicated slope segments, and extends to at least $t = 4.5$.

In part (b) students were told that $H(t) > 1$ and asked to find the value of t in $0 < t < 5$ at which H has a critical point. Then the students were asked to determine whether this critical point is the location of a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the seawater depth. A correct response would first determine that $\left.\frac{dH}{dt}\right|_{t=\pi} = 0$ and therefore the critical point in $0 < t < 5$ occurs when $t = \pi$. Because $\frac{dH}{dt}$ changes from positive to negative at $t = \pi$, this critical point is the location of a relative maximum value of H.

In part (c) students were asked to use the separation of variables technique to find an expression for the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$. A correct response will separate the variables H and t, integrate, use the initial condition to find the value of the constant of integration, and arrive at a solution of $H(t) = 1 + 3e^{\sin(t/2)}$.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

<u>Part (a)</u>

- Most of the responses correctly sketched a solution curve that passed through the point $(0, 4)$.
- Only a very small number of responses made no attempt to answer this part of the question.

<u>Part (b)</u>

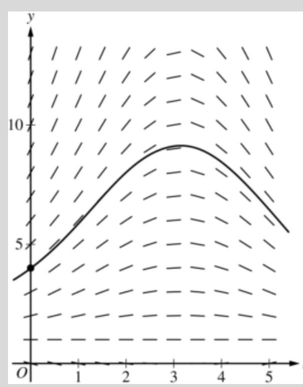
- Responses indicated that this part of the prompt was very approachable.
- The first point for considering the sign of $\frac{dH}{dt}$ was earned in a variety of ways. The most common response that earned this point was writing the equation $H'(t) = 0$. Many responses simply stated that $H'(t)$ changed signs.
- Many responses earned the second point by presenting $t = \pi$, although some responses instead reported $t = 3.14$ or $t = 3$, neither of which earned the second point as these values are not correct to three digits to the right of the decimal. Some responses included the value $t = 3\pi$ as a critical value, but because this value was not in the given interval $0 < t < 5$, presenting this value did not affect scoring. (Responses typically eventually discarded this value for the appropriate reason).

- Many responses earned the justification point with a statement such as "H has a relative maximum at $t = \pi$ because $H'(t)$ changes from positive to negative there.".
- Some responses used vague or incomplete language to justify the maximum, e.g., "the slope," "the graph of the slope field," or "the graph of H".

Part (c)

- Many responses were able to successfully separate the variables in this part. Some of the separations were incorrect only because of the position of the constant, e.g., $\int \frac{dH}{2(H-1)} = \int \cos\left(\frac{t}{2}\right) dt$.
- A large number of responses were able to find $\int \frac{dH}{H-1} = \ln|H-1|$, some with and some without the absolute value symbols. Presenting $\ln(H-1)$ instead of $\ln|H-1|$ did not affect scoring in this case but should not be overlooked.
- Some responses were able to correctly integrate and present $\int \frac{1}{2}\cos\left(\frac{t}{2}\right) dt = \sin\left(\frac{t}{2}\right)$, although many responses did not find the correct antiderivative and instead reported functions such as $-\sin\left(\frac{t}{2}\right)$, $-\frac{1}{4}\sin\left(\frac{t}{2}\right)$, or $\frac{1}{4}\sin\left(\frac{t}{2}\right)$.
- Nearly all responses that correctly separated the variables and presented an equation involving the declared antiderivatives included a constant of integration at the appropriate time.
- Some of the responses that included the constant of integration correctly used the initial condition by substituting $H = 4$ and $t = 0$ into their equation. However, using $t = 4$ and $H = 0$ was not an uncommon response.
- Few responses earned all five points on this part of the prompt.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses did not extend the solution curve all the way to $t = 4.5$.	• 
• In part (b) some responses attempted to justify a relative maximum occuring at $t = \pi$ by arguing that H changes from increasing to decreasing there. • Some responses attempted to justify the relative maximum at $t = \pi$ using the Second Derivative Test but did not use implicit differentiation or did not correctly apply the product rule when finding $\frac{d^2H}{dt^2}$.	• $H(t)$ has a relative maximum at $t = \pi$ because $H'(t)$ changes from positive to negative there. • $\frac{d^2H}{dt^2} = \frac{1}{2}(H-1)\left(-\frac{1}{2}\sin\left(\frac{t}{2}\right)\right) + \cos\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dH}{dt}$

• In part (c) some responses separated incorrectly, e.g., $\int \frac{dH}{2(H-1)} = \int \cos\left(\frac{t}{2}\right) dt.$	• $\int \frac{2 \cdot dH}{(H-1)} = \int \cos\left(\frac{t}{2}\right) dt$ or $\int \frac{dH}{(H-1)} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right) dt$
• Many responses incorrectly antidifferentiated $\cos\left(\frac{t}{2}\right).$	• $\int \cos\left(\frac{t}{2}\right) dt = 2\sin\left(\frac{t}{2}\right) + C$
• Some responses tried to solve their equation for $H(t)$ before using the initial condition, e.g., $H - 1 = e^{\sin(t/2)+C} \Rightarrow 4 = e^C + 1 \Rightarrow \ln 3 = C$ $H = 1 + \ln 3 \cdot e^{\sin(t/2)}$	• $H - 1 = e^{\sin(t/2)+C} \Rightarrow 4 = e^C + 1 \Rightarrow \ln 3 = C$ $H = 1 + e^{\sin(t/2)+\ln 3} = 1 + 3e^{\sin(t/2)}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers could provide students with practice sketching solution curves of various types and could emphasize how information given in the question can be used to create a mental picture of the solution curve.
- Teachers should hold their students to a high standard in justifying the type of critical value that is found. Language such as “the derivative changed from positive to negative” or “the graph changed from positive to negative” or “it changed from positive to negative” is too vague. Practice using concise, precise statements and notation such as “$\frac{dH}{dt}$ changed from positive to negative” would be invaluable.
- Teachers could provide several options (correct and incorrect) of “separated” separable differential equations and have students practice classifying the options as “correctly separated,” “poorly separated,” or “not separated”.
- In practicing solving differential equations, teachers could encourage students to use the initial condition when the constant of integration first appears. For example, once $\ln|H-1| = \sin\left(\frac{t}{2}\right) + C$ is written, students should then immediately use $H = 4$ and $t = 0$ to obtain $\ln|4-1| = \sin\left(\frac{0}{2}\right) + C.$ The combination of the two equations would have earned the response the fourth point in part (c) no matter what algebraic mistakes were made in attempting to find $H(t)$.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- In Topic 7.6 on AP Classroom, there are links to a lesson and student handout: *Applying Procedures for Separable Differential Equations and General Solutions.* These resources address suggestions offered above by the Chief Reader (CR). Although the first activity is not about sketching, it is an analytical version of the CR’s first suggestion about practicing sketching solution curves. The lesson and handout also include an error analysis activity, related to the CR’s third suggestion above.
- Student Practice resources for Unit 7 are excellent preparation for this question. Level 1 offers an appropriate introduction to the methods; Level 2 includes a slope field and advances the difficulty a bit; and Level 3 develops even deeper understanding. Level 3 looks back to the math of ratio and proportions learned before calculus, incorporates limits, and guides students through the particular solution of a challenging separable differential equation. These resources can be found on the Unit 7 page on AP Classroom.

Question AB4/BC4
Topic: Graphical Analysis with FTC
Max Score: 9
Mean Score AB4: 3.86
Mean Score BC4: 5.84

What were the responses to this question expected to demonstrate?

In this question the graph of a differentiable function f, for $-6 \le x \le 7$, and the shaded region R in the second quadrant bounded by the graph of f, the vertical line $x = -6$ and the x- and y-axes are shown. Students are told that f has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Students are also told that region R has area 12.

In part (a) the function $g(x) = \int_0^x f(t)\,dt$, is defined and students are asked to find the values $g(-6)$, $g(4)$, and $g(6)$. A correct response would recognize that $g(-6) = \int_0^{-6} f(t)\,dt = -(\text{area of } R) = -12$. In addition, $g(4) = \int_0^4 f(t)\,dt$ is the area of a triangle of base 4 and height 2, so $g(4) = 4$. Finally, $g(6) = g(4) + \int_4^6 f(t)\,dt$, where $\int_4^6 f(t)\,dt$ is the area of a triangle with base 2 and height -1. Thus, $g(6) = 4 - 1 = 3$. Throughout this part of the problem, students are asked to demonstrate knowledge of the properties of definite integrals.

In part (b) students are asked to find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. A correct response would recognize that by the Fundamental Theorem of Calculus, the derivative of the function g is the function f ($g' = f$), and therefore the critical points of g occur where $f(x) = 0$ or where $f(x)$ is undefined. Because $f(x)$ is differentiable, $f(x)$ is defined for all x in the interval $[0, 6]$. Therefore, $f(x) = 0 \Rightarrow x = 4$. Thus, the only critical point in this interval occurs at $x = 4$.

In part (c) a third function, h, is defined as $h(x) = \int_{-6}^x f'(t)\,dt$, and students are asked to evaluate this function and its first two derivatives at $x = 6$. A correct response will use the Fundamental Theorem of Calculus and the given graph of f to find that $h(6) = \int_{-6}^6 f'(t)\,dt = f(6) - f(-6) = -1.5$. Applying the Fundamental Theorem of Calculus again, a correct response would indicate that $h'(x) = f'(x)$ and use the fact that f is linear for $0 \le x \le 7$ to find that $h'(6) = f'(6)$ equals the slope of f at $x = 6$. Finally, a correct response would report that $h''(6) = f''(6) = 0$ because f is linear for $0 \le x \le 7$.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

Part (a)

- Most responses indicated at least a proficient understanding of the properties of definite integrals. These responses correctly used geometry to evaluate $g(4)$ and $g(6)$.
- The most common mistake in this part was in failing to recognize that $g(-6)$ was the *negative* of the area of region R.
- Some responses calculated the areas as if $g(x)$ were defined as $g(x) = \int_{-6}^x f(t)\,dt$, resulting in answers that were all too large by a value of 12.

Part (b)

- A majority of the responses correctly recognized and reported the connection that $g'(x) = f(x)$, which is not surprising as such a question has appeared on virtually all AP calculus exams in the past 5 years.
- Nearly all responses that applied the Fundamental Theorem of Calculus continued and successfully reported the sole critical point and gave a sufficient reason for their answer.
- Responses that did not earn full credit on this part tended to provide reasons that used vague language, such as "the graph" or "the slope", or that referenced the behavior of the function f without connecting the given graph of f to the derivative of the function g.

Part (c)

- A large number of responses correctly evaluated $h(6)$ by applying the Fundamental Theorem of Calculus (FTC) and reading the values $f(6)$ and $f(-6)$ from the graph.
- Some responses incorrectly applied the FTC, reporting that $h(6) = f(6) = -1$ or $h(6) = f'(6) - f'(-6)$.
- Many responses correctly found that $h'(6) = f'(6)$ and recognized that $f'(6)$ is the value of the slope of the graph of f at $x = 6$.
- Nearly all responses were able to correctly find $h''(6) = 0$.
- Quite a few responses incorrectly equated an expression with a specific value, e.g., $h'(x) = h'(6)$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a), many responses incorrectly reported $g(-6) = \int_0^{-6} f(t)\,dt = 12$.	• $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$
• Some responses included poor notation or communication, such as $g(-6) = \int_0^{x} f(-6)\,dt = -12$.	• $g(-6) = \int_0^{-6} f(t)\,dt = -12$
• Some responses presented incorrect expressions such as $\int_0^6 f(t)\,dt = \int_0^4 f(t)\,dt + \int_0^6 f(t)\,dt$	• $\int_0^6 f(t)\,dt = \int_0^4 f(t)\,dt + \int_4^6 f(t)\,dt$
• Some responses incorrectly applied the FTC, e.g., $\int_0^x f(t)\,dt = g(x) - g(0) = g(x) - 2$, perhaps because the given graph showed that $f(0) = 2$.	• $g(0) = \int_0^0 f(t)\,dt = 0$
• Some responses used $x = -6$ as the lower limit of integration: $g(-6) = \int_{-6}^{-6} f(t)\,dt = 0$, $g(4) = \int_{-6}^{4} f(t)\,dt = 16$, and/or $g(6) = \int_{-6}^{6} f(t)\,dt = 15$.	• $g(-6) = \int_0^{-6} f(t)\,dt = -12$, $g(4) = \int_0^4 f(t)\,dt = 4$, $g(6) = \int_0^6 f(t)\,dt = 3$

• In part (b) some responses reversed the relationship between f and g, reporting $f'(x) = g(x)$.	• $g'(x) = f(x)$
• Some responses used poor notation, e.g., $g'(x) = f(t)$.	• $g'(x) = f(x)$ or $g'(t) = f(t)$
• Several responses did not provide the support of $g' = f$ as part of the reason why $x = 4$ is the location of a critical point of g.	• $g(x)$ has a critical point at $x = 4$ because $g'(4) = f(4) = 0$
• Many responses used vague language such as "there is a critical point at 4 since it goes from positive to negative there.".	• $g(x)$ has a critical point at $x = 4$ because $g'(x)$ changes from positive to negative there.
• In part (c) some responses misconstrued the FTC, e.g., $\int_{-6}^{6} f'(t)\,dt = f(6) = -1$.	• $\int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5$
• Some responses incorrectly applied the FTC, e.g., $h(6) = \int_{-6}^{6} f'(t)\,dt = f'(6) - f'(-6)$.	• $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6)$
• Some responses reversed the limits of integration, e.g., $h(6) = \int_{-6}^{6} f'(t)\,dt = f(-6) - f(6)$.	• $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6)$
• Some responses equated unequal expressions, such as $h'(6) = f'(x) = f'(6)$ and/or $h''(6) = h''(x) = f''(x) = f''(6)$.	• $h'(x) = f'(x)$ therefore $h'(6) = f'(6)$, and $h''(x) = f''(x)$ therefore $h''(6) = f''(6)$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers could emphasize problems which vary the lower limit of integration for functions defined by an integral. In addition, teachers could assign more problems with the lower limit of integration being greater than the upper limit, so that students practice determining appropriate signs for definite integral values.
- Teachers could incorporate more problems that define one function as a derivative (or antiderivative) of another and then require that the students draw conclusions about one given information about the other. In particular, using more functions defined in terms of an integral could reinforce the Fundamental Theorem of Calculus.
- Teachers could provide students with opportunities to compare and contrast functions that may at first look similar. For example, compare the following: $\int_a^x f'(t)\,dt$, $\int_a^x f(t)\,dt$, $\frac{d}{dx}\int_a^x f'(t)\,dt$, and $\frac{d}{dx}\int_a^x f(t)\,dt$.
- Teachers could emphasize the importance of using correct and specific terminology ,e.g., saying "$g'(x)$" versus just saying "the graph" or "the derivative", and clearly distinguishing between the notation for a derivative function versus the value of the derivative at a specific point ,e.g., $h'(x)$ versus $h'(6)$. Encouraging students to carefully label all of their work may reduce the likelihood of their using terms such as "it" or "the function" in their reasoning or explanations.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

In Topic 6.5 on AP Classroom, please find a link to a lesson and student handout: *Justifying Behavior of* $f(x)$ *from a graph of* $g = f'(x)$.

- The first activity in this resource is similar to part (a) of this question and provides practice evaluating a function defined by an integral as described by the Chief Reader in her first suggestion above. When moving on to Topic 8.7, asking students to use a property of integrals to rewrite $g(-3) = \int_{-1}^{-3} f(t)\,dt$ so that the limits of integration go from smaller to greater values of x would help to reinforce students' understanding of the appropriate sign for such values.
- The next activities in this resource consider the graph and behaviors of a function $g(x) = \int_{-1}^{x} f(t)\,dt$ and build the foundational understanding needed in preparation for questions like part (b) of this question.
- The last activity in this resource asks students to complete a set of "sentence starters" that lay the groundwork for skill 4.A (*Use precise mathematical language*). As students build proficiency in this skill, they can practice creating sentence starters for themselves using language in the stem of the question. In part (b) of this question, they might have started by writing, "The graph of g has a critical point at x=_______, because _______.". A student who understands what a critical point is will start thinking about $g'(x)$. A student who also understands the Fundamental Theorem of Calculus might put the two thoughts together to write "$g'(x) = f(x) = 0$ there" in the second blank and is one easy step away from writing 4 in the first blank.

Question AB5

Topic: Implicit Differentiation with Related Rates
Max Score: 9
Mean Score: 3.14

What were the responses to this question expected to demonstrate?

This question began by asking students to consider the curve implicitly defined by the equation $x^2 + 3y + 2y^2 = 48$. Students were given that for this curve, $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

In part (a) students were asked to use the line tangent to the curve at the point $(2, 4)$ to approximate the y-coordinate of the point on the curve with x-coordinate 3. A correct response will use the given equation for $\frac{dy}{dx}$ to find the slope of this tangent line and then use the equation of the tangent line at $(2, 4)$, $y = 4 - \frac{4}{19}(x - 2)$, to approximate the y-coordinate when $x = 3$.

In part (b) students were asked to determine whether the horizontal line $y = 1$ is tangent to the implicitly defined curve. A correct response will recognize that a horizontal tangent line must have a slope of 0, and therefore must satisfy $\frac{dy}{dx} = 0$. Therefore, a the horizontal line $y = 1$ must pass through the point $(0, 1)$. The response will then find that $0^2 + 3 \cdot 1 + 2 \cdot 1^2 \neq 48$, so the line $y = 1$ cannot be tangent to the given curve.

Alternatively, a correct response could use $y = 1$ and the equation of the curve to find $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48$, which happens only when $x = \pm\sqrt{43}$. However, at either point $\left(\pm\sqrt{43}, 1\right)$, the slope $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$ is not 0, so the line $y = 1$ cannot be tangent to the given curve.

In part (c) students were told that the curve intersects the x-axis at the point $\left(\sqrt{48}, 0\right)$ and asked whether the line tangent to the curve at this point is vertical. A correct response will find that at this point on the curve the slope of the tangent line is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3 + 4 \cdot 0}$, with denominator $3 + 4 \cdot 0 \neq 0$. Thus, the line tangent to the curve at the point $\left(\sqrt{48}, 0\right)$ is not vertical.

In part (d) students were given a different equation, $y^3 + 2xy = 24$, and were told that on the curve implicitly defined by this equation, at the instant when the particle is at the point $(4, 2)$, the y-coordinate of the particle's position was decreasing at a rate of 2 units per second. Students were asked to find the rate of change of the x-coordinate of the particle's position with respect to time at that instant. A correct response will implicitly differentiate the equation with respect to t, use $x = 4$, $y = 2$, and $\frac{dy}{dt} = -2$ in the resulting differentiated equation, and solve for $\frac{dx}{dt}$.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

Part (a)

- Most responses that attempted this part were able to find the slope of the tangent line at the point $(2, 4)$.
- Most of the responses that found the slope of the tangent line also successfully wrote an equation for the tangent line and used $x = 3$ in the equation to find an approximation. However, many of these responses attempted to simplify their approximations and made arithmetic errors in their simplifications.

Part (b)

- A majority of the responses began this problem by substituting $y = 1$ into the equation of the curve, then solved for the corresponding values of x. Such responses usually went on to try to demonstrate that the slope of the curve at the resulting point(s) was not zero.
- However, many of these responses only found the point $(\sqrt{43}, 1)$, failing to consider $(-\sqrt{43}, 1)$, the other point on the curve with $y = 1$.
- Some responses did begin the problem by setting $\frac{dy}{dx} = 0$, and these responses easily determined that this resulted in $x = 0$. Unfortunately, such responses were rarely able to correctly reason that the point $(0, 1)$ was not on the given curve.

Part (c)

- A majority of the responses noted that the slope of a vertical line is undefined, and went on to correctly calculate the slope of the line tangent to the curve at the point $(\sqrt{48}, 0)$.
- Unfortunately, many responses struggled to provide correct verbal reasoning that the tangent line at this point could not be vertical.

Part (d)

- Many responses recognized this part as a "related rates" problem and recognized the need to differentiate the given equation with respect to t. Such responses generally performed very well on this part, earning at least three of the four points.
- Some responses chose to differentiate the given equation with respect to x, then solve for $\frac{dy}{dx}$. Such responses needed to continue by using the chain rule to find $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, then use $\frac{dy}{dt} = -2$ and $\frac{dy}{dx} = -\frac{1}{5}$ to find $\frac{dx}{dt}$. Many of the responses that did begin the problem by differentiating with respect to x did not provide any work beyond this step.
- The majority of responses that differentiated the equation $y^3 + 2xy = 24$ with respect to t or with respect to x did correctly apply the product rule to differentiate the term $2xy$.
- Quite a few responses struggled with correct mathematical notation for differentials, and several confused the values of $\frac{dy}{dt}$ and $\frac{dy}{dx}$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) responses were sometimes uncertain whether to use $x = 2$ or $x = 3$ to find the slope and/or equation of the tangent line. • Several responses made arithmetic mistakes when attempting to simplify tangent line approximations.	• $\frac{dy}{dx} = \frac{-2 \cdot 2}{3 + 4 \cdot 4} = -\frac{4}{19}$ • $y = \left(\frac{-2 \cdot 2}{3 + 4 \cdot 4}\right) \cdot (3 - 2) + 4$ requires no simplification
• In part (b) many responses began by using $y = 1$ in the given curve and found only $x = +\sqrt{43}$. Often responses that did solve and find both roots, $x = \pm\sqrt{43}$, did not determine that the slope of the line through $\left(-\sqrt{43}, 1\right)$ was nonzero. • Some responses that found the points $\left(\pm\sqrt{43}, 1\right)$ argued that the line $y = 1$ could not be tangent to the curve because "tangent lines cannot touch a graph twice." • Several responses attempt to use only nonnumerical arguments to reason that the line $y = 1$ is not tangent to the curve.	• $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48 \Rightarrow x = \pm\sqrt{43}$ $\frac{dy}{dx} = \pm\frac{2\sqrt{43}}{3 + 4 \cdot 1} \neq 0$, which means the horizontal line at $y = 1$ cannot be tangent to the curve. • A horizontal line has slope $= 0$. $\frac{dy}{dx} = \frac{-2x}{3 + 4y} = 0$ only when $x = 0$. Thus, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$. However, $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48$, so $y = 1$ is not tangent to the curve.
• In part (c) many responses included poor or incorrect communication. For example, many responses included the statements "the slope is vertical" or "the tangent line is undefined." • Some responses began with a statement that in order for the tangent line to be vertical, $\frac{dy}{dx}$ would need to be undefined, but concluded with a statement that $\frac{dy}{dx} \neq 0$.	• At the point $\left(\sqrt{48}, 0\right)$ the slope of the tangent line is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3 + 4 \cdot 0}$, which is defined. Therefore, the line tangent to the curve at this point cannot be vertical. • The slope of the tangent line, $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$, is undefined when $3 + 4y = 0$, which occurs when $y = -\frac{3}{4} \neq 0$. Thus, the tangent line through $\left(\sqrt{48}, 0\right)$ cannot be vertical.

• Many responses did not connect "decreasing" with a negative sign on $\frac{dy}{dt}$ and used $\frac{dy}{dt} = +2.$	• At the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second $\Rightarrow \frac{dy}{dt} = -2.$
• In part (d) some responses did not recognize the need to differentiate with respect to t, and therefore began by differentiating with respect to x. Although some of these responses did correctly determine that $\frac{dy}{dx} = -\frac{1}{5}$, they often incorrectly used $dy = -2$, and presented a solution of $\frac{dy}{-\frac{1}{5}} = dx \Rightarrow dx = (-2)\cdot(-5) = 10.$	• $3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$ $\frac{dy}{dx} = \frac{-2y}{\left(3y^2 + 2x\right)} \Rightarrow \left.\frac{dy}{dx}\right\|_{(x,\, y)=(4,\, 2)} = \frac{-2\cdot 2}{\left(3\cdot 2^2 + 2\cdot 4\right)} = -\frac{1}{5}$ $\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx} \Rightarrow \frac{dx}{dt} = -2 \div \left(-\frac{1}{5}\right) = 10$
• Quite a few responses confused differentials with derivatives, writing dy and dx when they should have written $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Other responses presented equations containing an incorrect mix of $\frac{dy}{dt}$, y', dy, and/or $\frac{dy}{dx}$.	• $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$ $\frac{dy}{dt} = -2$ $3\cdot 2^2(-2) + 2\cdot 4(-2) + 2\cdot 2\frac{dx}{dt} = 0$ $\frac{dx}{dt} = \frac{40}{4} = 10$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers could emphasize the importance of proper notation and the value of good communication. In particular, when students are given a result and asked to show that it is true, the students must make sure that the work shown is accurate and supports the claim. If students observe the need to make an adjustment because the work does not result in the desired answer, the students should backtrack and find where the error originated. All of the steps in the problem must be consistent in leading to the given answer.
- Teachers could have their students frequently write one to two sentence justifications for answers and in the process help students connect the correct nouns and adjectives. Teachers could have students practice using concise and specific language, avoiding vague words such as "it." Accurate communication is part of what is tested on the exam, and students need more practice and feedback in this area. It might be helpful to provide students with a prompt and an answer, then ask students to explain whether the supplied answer is correct.
- Teachers could emphasize that proper calculus notation is important. Students should know the differences between notations such as $\frac{dy}{dt}$, dy, y', and $\frac{dy}{dx}$, as well as which notation is appropriate in various situations. It could also be discussed that, although it may be correct, notation such as x' or y' does not always provide clear communication of the variable of differentiation.
- Teachers could provide more related rates problems that are not context based but stem from an implicit relationship between x and y. Teachers could provide practice with problems involving implicit differentiation that go beyond merely finding $\frac{dy}{dx}$. It would be helpful to provide graphs of implicitly defined functions so that students could visualize implicitly defined curves and their tangent lines.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- Addressing the Chief Reader's suggestions regarding the importance of the skillful use of appropriate mathematical symbols and notation (skill 4.C) in this question may begin with careful attention to detail to the relevant variables when learning the chain rule and implicit differentiation in topics 3.1 and 3.2. Topic 3.2 on AP Classroom provides lessons and student handouts to help establish a strong foundation. *Applying the Chain Rule for Derivatives of Composite Functions* begins by developing basic understanding, provides practice using the chain rule, and concludes with an activity to help students to sort out when the chain rule is needed in situations including implicit differentiation. This handout models careful use of notation and can be used to emphasize that skill.
- Unit 3 Student Practice Level 1, approaches issues relevant to this question at an entry level, especially part (c). Unit 3 Student Practice Level 2 takes students through their paces at a slightly higher level. Part (d) focuses on implicit differentiation. These resources can be found on the Unit 3 page of AP Classroom.

Question AB6

Topic: Area-Volume
Max Score: 9
Mean Score: 3.59

What were the responses to this question expected to demonstrate?

In this problem a graph of functions $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$ for $0 \le x \le 5$ was shown.

In part (a) students were told that the shaded region R is bounded by the graphs of f and g, from $x = 0$ to $x = 2$. Students were asked to write, but not evaluate, an integral expression that gives the area of region R. A correct response will provide the integral $\int_0^2 (f(x) - g(x))\, dx$, including the outer parentheses and differential.

In part (b) students are told that the shaded region S is bounded by the graph of g and the x-axis, from $x = 2$ to $x = 5$, and that this region is the base of a solid where for each x the cross section perpendicular to the x-axis is a rectangle with height equal to half its base in region S. Students were asked to find the volume of this solid. A correct response will present the definite integral $\int_2^5 \frac{1}{2}(g(x))^2\, dx = \int_2^5 \frac{1}{2}\left(x^2 - 2x\right)^2 dx$, recognize the need to expand $(g(x))^2$, find an antiderivative for $\frac{1}{2}\left(x^4 - 4x^3 + 4x^2\right)$, and compute the difference of the antiderivative evaluated at $x = 5$ and at $x = 2$.

In part (c) students were asked to write, but not evaluate, an integral expression that gives the volume of the solid that is generated when the region S is rotated about the horizontal line $y = 20$. A correct answer will present the definite integral $\pi\int_2^5 \left[20^2 - (20 - g(x))^2\right] dx$, including the outer brackets or parentheses and the differential.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

Part (a)

- The vast majority of the responses presented an integrand containing a difference involving the functions $f(x)$ and $g(x)$, thus earning the first point in this part.
- Nearly all of the responses that earned the first point also earned the second point for the correct integral, although some of these integrals were missing the differential dx.
- Several responses chose to write the given expressions for the functions, but neglected to use sufficient parentheses, resulting in an incorrect integrand.
- A small number of responses presented integrals with incorrect limits of integration or presented an integral expression for the area of region S, thus failing to earn both points in this part.

Part (b)

- A majority of the responses earned at least one point in this part, usually for the limits of integration in a definite integral with an integrand of the form $a(x) \cdot g(x)$.
- Responses that recognized the need to expand the expression $\left(x^2 - 2x\right)^2$ before integrating were generally successful in providing a correct antiderivative.
- Most responses that provided an antiderivative for their integrand demonstrated an understanding of the Fundamental Theorem of Calculus by correctly evaluating the antiderivative at the limits of integration. However, some of the numerical answers were simplified incorrectly.

Part (c)

- Students struggled to provide correct answers in this part of the problem.
- Some responses did present the correct form of the integrand but reversed the terms, e.g., $\left[(20 - g(x))^2 - \left(20^2\right)\right]$.
- Some responses initially presented the correct integrand but chose to continue to write the explicit expression for $g(x)$ and failed to include the necessary parentheses. For example, $\left[\left(20^2\right) - \left(20 - x^2 - 2x\right)^2\right]$.
- A few responses presented a correct integrand and limits of integration but failed to include the constant π.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses initially wrote a correct expression in terms of $f(x)$ and $g(x)$ but subsequently presented an error in using the given expressions for these functions. For example, $\int_0^2 (f(x) - g(x))\,dx = \int_0^2 \left(x^2 + 2 - x^2 - 2x\right) dx$ $= \int_0^2 (2 - 2x)\,dx.$	• A response of $\int_0^2 (f(x) - g(x))\,dx$ is sufficient. If a response does use the given expressions for $f(x)$ and $g(x)$, parentheses are necessary. For example, $\int_0^2 (f(x) - g(x))\,dx = \int_0^2 \left(x^2 + 2 - \left(x^2 - 2x\right)\right) dx$ $= \int_0^2 (2 + 2x)\,dx.$
• Several responses failed to include parentheses or a differential as part of their integrand, e.g., $\int_0^2 f(x) - g(x).$	• $\int_0^2 (f(x) - g(x))\,dx$
• In part (b) many responses had difficulty translating the description of the solid provided in the prompt to an appropriate integrand, e.g., $\int_2^5 \frac{1}{2} g(x)\,dx$ or $\int_2^5 2(g(x))^2\,dx.$	• $\int_2^5 \frac{1}{2}(g(x))^2\,dx$
• Some responses presented algebraic errors when expanding the expression $\left(x^2 - 2x\right)^2$.	• $\left(x^2 - 2x\right)^2 = x^4 - 4x^3 + 4x^2$
• Many responses attempted to integrate $\left(x^2 - 2x\right)^2$ without first expanding the expression, obtaining an antiderivative of $\frac{\left(x^2 - 2x\right)^3}{3}$.	• $\frac{1}{2}\int_2^5 \left(x^2 - 2x\right)^2 dx = \frac{1}{2}\int_2^5 \left(x^4 - 4x^3 + 4x^2\right) dx$ $= \frac{1}{2}\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_2^5$
• Too many responses indicated a significant misconception of the properties of definite integrals by rewriting their integrals as $\frac{1}{2}\int_2^5 g(x)\,dx \cdot \int_2^5 g(x)\,dx.$	• $\frac{1}{2}\int_2^5 \left(x^2 - 2x\right)^2 dx = \frac{1}{2}\int_2^5 \left(x^4 - 4x^3 + 4x^2\right) dx$

• Some responses unsuccessfully attempted to use the washer method to find the requested volume. • Some responses made arithmetic errors in attempting to simplify their numeric answer.	• $\frac{1}{2}\left[\left(\frac{5^5}{5}-5^4+\frac{4\cdot 5^3}{3}\right)-\left(\frac{2^5}{2}-2^4+\frac{4\cdot 2^3}{3}\right)\right]$ is sufficient, but simplifies to $\frac{1}{2}\left(\frac{500}{3}-\frac{16}{15}\right)=\frac{414}{5}$.
• In part (c) many responses presented an incorrect form of the integrand, e.g., $20^2-(g(x))^2$ or $(20-g(x))^2$. • Some responses reversed the inner and outer radii resulting in an integrand of $\left[(20-g(x))^2-\left(20^2\right)\right]$. • Some responses initially presented a correct integral expression $\pi\int_2^5\left[20^2-(20-g(x))^2\right]dx$, but subsequently presented an error in the given expression for $g(x)$, e.g., $\pi\int_2^5\left[20^2-\left(20-x^2-2x\right)^2\right]dx$. • A few responses failed to include the constant π or a differential in their integral expression.	• $\pi\int_2^5\left[20^2-(20-g(x))^2\right]dx$ • $\pi\int_2^5\left[400-(20-g(x))^2\right]dx$ • $\pi\int_2^5\left[20^2-\left(20-\left(x^2-2x\right)\right)^2\right]dx$ • $\pi\int_2^5\left[400-(20-g(x))^2\right]dx$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Students need a better understanding that various forms are acceptable when presenting answers. In particular, when functions $f(x)$ and/or $g(x)$ are defined in the stem, it is sufficient to write setups that use only the symbols f and/or g; it is not necessary to copy the expressions given for $f(x)$ and $g(x)$.
- Teachers could provide students more opportunities to practice with integrands that require expanding or algebraic rewriting and/or algebraic manipulation before attempting to find an antiderivative.
- Students need more opportunities to develop their conceptual understanding of volumes involving rotation and cross sections. Teachers could implement hands-on activities, computer simulations, and investigations that allow students to better visualize how these solids are formed.
- Teachers could reinforce proper communication in student work by emphasizing the importance of parentheses to group expressions and the differential in an integral.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- Student Practice Level 2 found on the Unit 8 page on AP Classroom is an excellent resource for practicing setting up common area and volume solutions and finding answers using a calculator. Also found on the Unit 8 page on AP Classroom, Student Practice Level 3 is a resource for practicing solving area volume questions without the use of a calculator. Part (a) of the Level 3 resource includes evaluating an integral that requires squaring a binomial, as in this question. As a bonus, part (b) also considers the solution that sets up an evaluates an integral with respect to y to find the same value.

Question BC2

Topic: Parametric Particle Motion – Speed, Distance, Position
Max Score: 9
Mean Score: 5.56

What were the responses to this question expected to demonstrate?

In this question students were told a particle was moving along a curve in the xy-plane with position $(x(t), y(t))$ at time t seconds. Students were also given expressions for $x'(t)$ and $y'(t)$ and were told that particle was at the point $(3, 6)$ at time $t = 2$.

In part (a) students were asked to find the speed of the particle at time $t = 2$ seconds. A correct response would provide the setup $\sqrt{(x'(2))^2 + (y'(2))^2}$ and use a calculator to find the numerical value 12.305 (or 12.304).

In part (b) students were asked to find the total distance traveled by the particle over the time interval $0 \le t \le 2$. A correct response would present the setup $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ and use a calculator to find the value 15.902 (or 15.901).

In part (c) students were asked to find the y-coordinate of the position of the particle at time $t = 0$. A correct response would provide the setup $y(0) = 6 + \int_2^0 y'(t)\, dt$ and use a calculator to find the value -1.174 (or -1.173).

In part (d) students were told that the particle remains in the first quadrant during the time interval $2 \le t \le 8$ and were asked to find all times in this interval when the particle is moving toward the x-axis. A correct response would consider the sign of $y'(t)$ and answer that because $y'(t) < 0$ on this time interval, the particle will be moving toward the x-axis.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

<u>Part (a)</u>

- Most responses provided the correct setup for the speed of the particle.
- Most responses also successfully used a calculator to determine the speed of the particle at time $t = 2$.
- A few of the responses included absolute value symbols either inside or outside the square root. Although this was not incorrect, it may have suggested a weaker understanding of the expression for finding the speed of the particle moving on the plane or about the meaning of the square root function.
- Some responses presented fewer than three correct digits after the decimal, and those that did tended to do so in all parts of this problem.
- Some responses incorrectly equated a general expression in t to the value of such an expression at a particular point, e.g., $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(x'(2))^2 + (y'(2))^2}$.
- Frequently responses that chose to write the expression for speed using the given numeric expressions for $x'(t)$ and $y'(t)$ presented copy or presentation errors, such as dropping a square on one term or failing to include necessary parentheses.

Part (b)

- Nearly all responses presented a correct expression for the total distance traveled by the particle as $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt,$ although sometimes there was no differential included.
- Although units were not required, many responses did include units, and most of those responses reported correct units.
- Most responses that presented a correct setup also correctly evaluated the integral using a calculator.
- Although some responses presented an incorrect setup of $\int_0^2 \sqrt{(x'(2))^2 + (y'(2))^2}\, dt,$ some of these responses did evaluate the correct integral using their calculator.
- As in part (a), responses that chose to write the total distance expression using the given expressions for $x'(t)$ and $y'(t)$ frequently contained copy or presentation errors, such as failing to include necessary parentheses.
- As in part (a), some responses included absolute value symbols either inside or outside the square root.

Part (c)

- Many responses were able to correctly provide the setup and find the value of $y(0)$.
- Some of the responses that did not earn full credit provided an understanding of the need for the Fundamental Theorem of Calculus and presented an integral with integrand $y'(t)$ and limits of integration $t = 0$ and $t = 2$.
- A noticeable percentage of these responses reversed the limits of integration, reporting a positive integral value and an incorrect answer of $y(0) = 13.174$.
- A large number of the responses that presented a correct definite integral also incorporated the initial condition, $y(2) = 6,$ in some manner.
- Several responses that presented a definite integral did not include a differential as part of the integral. When this resulted in an ambiguously presented setup, such as $y(0) = \int_2^0 y'(t) + y(2),$ the response did not earn all of the points it might otherwise have earned.
- A few responses provided a correct setup and value for the particle's x-coordinate at time $t = 0,$ indicating a good understanding of the concepts tested by the question but a failure to carefully read the question asked.

Part (d)

- Many responses did consider the sign of $y'(t)$ in some way. Notation for $y'(t)$ was quite varied, including $y',$ $\frac{dy}{dt},$ and $dy.$
- Only some of the responses connected the particle moving toward the x-axis to $y'(t) < 0.$
- Some responses provided the interval $5.22 < t < 8,$ which contains a decimal presentation error. Frequently such responses had already presented a value with fewer than three correct digits after the decimal in a previous part of the problem, in which case such presentation did not affect their score in part (d).

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses defined the speed of the particle as the absolute value of $\frac{dy}{dx}$. • Many responses inappropriately equated a variable expression to a numerical value, e.g., $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(x'(2))^2 + (y'(2))^2}$. • A few responses presented an incorrect setup and answer of $\text{speed} = \frac{y''(2)}{x''(2)} = -0.214$. • Some responses presented a speed expression of $\sqrt{(x'(t))^2 + (y'(t))^2}\ dt$, indicating a misconception about the role of the differential.	• $\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$ • $\text{Speed at time } 2 = \sqrt{(x'(2))^2 + (y'(2))^2}$ • $\text{Speed at time } 2 \text{ is } \sqrt{(x'(2))^2 + (y'(2))^2} = 12.305$ • $\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$
• In part (b) some responses reported the total distance traveled as $\sqrt{\left(\int_0^2 x'(t)^2\,dt\right)^2 + \left(\int_0^2 y'(t)^2\,dt\right)^2}$. • Some responses presented a setup of $\int_0^2 \sqrt{(x'(2))^2 + (y'(2))^2}\ dt$, although they often evaluated the correct expression using a calculator. • Some responses presented incorrect setups such as $\int_0^2 \sqrt{(x'(2))^2 - (y'(2))^2}\ dt$ or $\int_0^2 \left(\frac{y'(t)}{x'(t)}\right) dt$.	• $\text{Total distance traveled} = \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$ $= 15.902$
• In part (c) many responses attempted to find a closed-form antiderivative of $y'(t)$ instead of using a calculator to numerically evaluate $\int_2^0 y'(t)\,dt$. • Some responses used the initial condition incorrectly, e.g., $y(0) = \int_2^0 y'(t)\,dt - y(2)$ or $y(0) = \int_2^0 y'(t)\,dt - 6$. • Responses frequently did not include a differential; some of these resulted in ambiguous expressions, e.g., $y(0) = \int_2^0 y'(t) + y(2)$. • Some responses used incorrect limits of integration such as $\int_0^3 y'(t)\,dt$. These errors may have resulted from confusion about the initial condition $(3, 6)$.	• $\int_2^0 y'(t)\,dt = -7.1736$ • $y(0) = y(2) + \int_2^0 y'(t)\,dt = 6 + \int_2^0 y'(t)\,dt$ • $y(0) = \int_2^0 y'(t)\,dt + y(2)$ • $y(0) = y(2) - \int_0^2 y'(t)\,dt$

• In part (d) many responses referenced the sign of $\frac{dy}{dx}$ rather than the sign of $\frac{dy}{dt}$. For example, "the particle would be moving toward the x-axis only when $\frac{dy}{dx}$ is decreasing which is when $\frac{d^2y}{dx^2} < 0$."	• The particle will be moving toward the x-axis when $y'(t) < 0$.
• Some responses did report the correct interval but did not provide correct reasoning. Instead of indicating that $y' < 0$ on the interval, they made statements such as "y is decreasing."	• Because the particle is in the first quadrant, $y(t) > 0$, so the particle is moving toward the x-axis when $y'(t) < 0$. This occurs when $5.222 < t < 8$.
• Some responses never presented any clear consideration of the y-coordinate of the particle, instead presenting an answer of just "$5.222 < x < 8$."	• The particle is moving toward the x-axis when $5.222 < t < 8$ because this is when $y'(t) < 0$.
• Some responses never found an interval because they considered only $x'(t) < 0$.	• Because the particle is in the first quadrant, $y(t) > 0$, so the particle is moving toward the x-axis when $y'(t) < 0$. This occurs when $5.222 < t < 8$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers should remind students to be careful with the use of the equal sign; it is not punctuation. Inappropriately equating variable expressions with numerical values is a costly error. However, failing to ever use an equal sign may make a final answer unclear. The equal sign should be used whenever it is appropriate but should not be used to connect unequal quantities.
- Teachers should consistently model proper notation and always require it of their students.
- To help students to avoid decimal presentation errors, teachers should require students to save intermediate calculated values in their calculators and to present their answers correct to three places after the decimal throughout the school year.
- Teachers should remind students that the calculator may be essential on some parts of a calculator-active free response question. Students should be prepared to use their calculator on all parts of such a question and should clearly indicate the setup for all calculator usage.
- Some students seem to be unaware they are expected to communicate how they are utilizing their graphing calculator by showing the necessary setup prior to presenting numerical answers. This setup requires proper notation, and responses that lack notation or include incorrect notation are rarely eligible to earn all points. Teachers could require similar setups on all submitted work in their classrooms to ensure that this becomes reflexive for students.
- If a student provides work that is incorrect or leads nowhere and eventually resorts to using a calculator, the incorrect work should be crossed out, leaving only the setup that was used in the calculator.
- Although most students seemed to know how to use their calculators to perform most of the required operations, there is some evidence from the responses in part (d) that students did not know how to find the zero for a function with the necessary three digits after the decimal accurately. This may be a result of using the TRACE function rather than using the calculator to solve an equation, either by finding the intersection of two curves or using CAS features. Teachers could have students practice finding solutions of equations to at least three decimal places.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- An excellent resource for developing understanding of and proficiency with the content and skills assessed in this question is *Vectors: A Curriculum Module for AP® Calculus BC*, by Nancy Stephenson. This module was published in 2010 and is still highly valued today for its clear explanations and excellent practice problems. Days 4-6 of this module are especially relevant to this question.

Question BC5

Topic: Analysis of Functions – Arc Length, Euler's Method, Integration by Parts
Max Score: 9
Mean Score: 5.92

What were the responses to this question expected to demonstrate?

In this question, students were told that the function f is twice differentiable and that $f(0) = 0$. A table of selected values of x and $f'(x)$ was provided.

In part (a) students were told that for $x \geq 0,$ the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$ and were asked to find the value of $h'(\pi)$. A correct response will use the Fundamental Theorem of Calculus to find $h'(x) = \sqrt{1 + (f'(x))^2}$, then will evaluate this expression at $x = \pi$.

In part (b) students were asked what information the expression $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provides about the graph of f. A correct response indicates that this is the expression for the arc length of the graph of f on the interval $[0, \pi]$.

In part (c) students were told to use Euler's method to approximate $f(2\pi)$, starting at $x = 0$, with two steps of equal size. A correct response will use the line tangent to the graph of f at $(0, 0)$ to find an approximation for $f(\pi)$, then use the line tangent to the graph of f at $(\pi, f(\pi))$ to approximate the value of $f(2\pi)$.

In part (d) students were asked to find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. A correct response will use the technique of integration by parts to find an answer of $4(t + 5) \cdot \sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

Part (a)

- Most of the responses demonstrated a basic understanding of the concept of the derivative of an integral expression, applying the Fundamental Theorem of Calculus correctly to find $h'(x) = \sqrt{1 + (f'(x))^2}$.
- Nearly all responses that correctly applied the FTC went on to successfully evaluate their expression at $x = \pi$.

Part (b)

- Most responses recognized the given expression as an arc length or as a length.
- Some responses said the expression represented a "distance" which was too vague, unless the distance was explicitly connected to the graph of f.
- Quite a few responses gave a complete answer that included information about the interval $[0, \pi]$.
- Many responses did not include any reference to the interval $[0, \pi]$.

Part (c)

- Nearly all responses that began Euler's method correctly found a correct approximation of $f(2\pi)$.
- Roughly half of the responses attempted to use a linear approximation approach; the other half used a tabular approach.

- Some responses used $f(0) = 5$ instead of $f(0) = 0$ in the initial step but were able to otherwise present two steps of Euler's method correctly.

Part (d)

- A majority of the responses recognized the need to use integration by parts and most of these made appropriate choices for u and dv. A significant number of responses used a tabular approach to integration by parts.
- Most responses had no trouble integrating $t + 5$, although necessary parentheses were occasionally omitted.
- Some responses presented an incorrect sign on the antiderivative of $\cos\left(\frac{t}{4}\right)$ or used $\int u\, dv = uv + \int v\, du$.
- Some responses began by using a u-substitution for $\frac{t}{4}$ and then implemented integration by parts, but became confused about which function was the "u" from u-substitution and which function was the "u" from integration by parts.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses used two distinct variables in a single variable equation. For example: $h'(x) = \sqrt{1 + (f'(t))^2}$.	• $h'(x) = \sqrt{1 + (f'(x))^2}$
• Many responses equated a variable expression to a numerical value, e.g., $h'(\pi) = \sqrt{1 + (f'(x))^2}$, or $h'(x) = \sqrt{1 + 6^2}$.	• $h'(x) = \sqrt{1 + (f'(x))^2} \Rightarrow h'(\pi) = \sqrt{1 + (f'(\pi))^2}$
• Several responses applied the Fundamental Theorem of Calculus incorrectly, e.g., $h'(\pi) = \sqrt{1 + (f'(x))^2}\Big\vert_0^{\pi} = \sqrt{37} - \sqrt{26}$.	• $h'(\pi) = \sqrt{1 + (f'(x))^2}\Big\vert_{x=\pi} = \sqrt{37}$
• Quite a few responses replaced the variable x with the constant π before differentiating, e.g., $h'(\pi) = \frac{d}{dx}\left[\int_0^{\pi} \sqrt{1 + (f'(t))^2}\, dt\right]$.	• $h'(x) = \frac{d}{dx}\left[\int_0^{x} \sqrt{1 + (f'(t))^2}\, dt\right] = \sqrt{1 + (f'(x))^2}$ $h'(\pi) = \sqrt{1 + (f'(\pi))^2} = \sqrt{1 + 6^2}$

• In part (b) many responses were incomplete, stating that the given integral was "the arc length of f," with no mention of the interval $[0, \pi]$.	• $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the arc length of the graph of f on $[0, \pi]$.
• Some responses used ambiguous language such as "this represents the distance from 0 to π."	• $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the distance between 0 and π along the curve f.
• Some responses presented the interval incorrectly as $[\pi, 0]$.	• $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the arc length of the graph of f on $[0, \pi]$.
• Some responses indicated that the expression represented the area under f or that the expression provided some information about the value, slope, concavity, or continuity of f.	• $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ is the arc length of the graph of f on $[0, \pi]$.
• In part (c) some responses did not use the correct initial value, e.g., $f(\pi) \approx f(0) + \pi \cdot f'(0) = 5 + \pi \cdot 5$.	• $f(\pi) \approx f(0) + \pi \cdot f'(0) = 0 + \pi \cdot 5 = 5\pi$
• Some responses used an incorrect value of $f'(x)$ in one or more steps of Euler's method, e.g., $f(\pi) \approx f(0) + \pi \cdot f'(\pi) = 0 + \pi \cdot 6$.	• $f(\pi) \approx f(0) + \pi \cdot f'(0) = 0 + \pi \cdot 5 = 5\pi$
• Some responses provided a third step in using Euler's metho, e.g., $f(2\pi) \approx 11\pi + \pi \cdot f'(2\pi) = 11\pi + \pi \cdot 0$.	• Use exactly two steps: $f(\pi) \approx f(0) + \pi \cdot f'(0) = 5\pi$ $f(2\pi) \approx f(\pi) + \pi \cdot f'(\pi) = 5\pi + 6\pi = 11\pi$
• In part (d) several responses presented incorrect choices of $u = \cos\left(\frac{t}{4}\right)$ and $dv = t + 5$.	• $u = t + 5,\ dv = \cos\left(\frac{t}{4}\right)$
• Some responses did not find a correct antiderivative for $dv = \cos\left(\frac{t}{4}\right)$, incorrectly finding $v = \frac{1}{4}\sin\left(\frac{t}{4}\right)$ or $v = -4\sin\left(\frac{t}{4}\right)$.	• $\int \cos\left(\frac{t}{4}\right) dt = 4\sin\left(\frac{t}{4}\right)$
• Some responses did not correctly use integration by parts, e.g., $\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\cdot\sin\left(\frac{t}{4}\right) + \int 4\sin\left(\frac{t}{4}\right) dt$.	• $\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\cdot\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt$
• Some responses ommitted parentheses, e.g., $t + 5\sin\left(\frac{t}{4}\right)\cdot 4 - \int 4\sin\left(\frac{t}{4}\right) dt$.	• $(t+5)\sin\left(\frac{t}{4}\right)\cdot 4 - \int 4\sin\left(\frac{t}{4}\right) dt$
• Some responses presented incomplete parentheses, e.g., $4(t+5)\cdot\left(\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt\right.$.	• $4(t+5)\cdot\left(\sin\left(\frac{t}{4}\right)\right) - \int 4\sin\left(\frac{t}{4}\right) dt$
	• $\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\cdot\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C$
	• Functions f and h are not involved in part (d).

• Some responses failed to include a constant of integration: $\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\cdot\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right).$ • Some responses attempted to solve for the constant of integration by using information about f or h.	

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Teachers could provide more practice with the Fundamental Theorem of Calculus, in particular working with differentiating an integral expression with function limits. Teachers could require correct notation in each of these practice opportunities.
- Teachers could compare and contrast the formulas for arc length (Topic 8.13 and 9.3) for both single variable equations and parametric equations.
- Teachers could provide practice using clear communication of the steps and labels in Euler's method, particularly when using a tabular presentation.
- Teachers should look for ways to help students recognize situations when a particular technique of integration (substitution, by parts, partial fractions, etc.) is appropriate.
- Teachers could provide significant practice integrating by parts, expecting a clear delineation of both u and dv. Students should be required to provide unambiguous communication, including appropriate labels on intermediate steps, particularly if they are using tabular integration by parts.
- Teachers could provide practice with using both u-substitution and integration by parts within a single integration problem.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The Daily Videos for Topic 6.11 and 6.14 in AP Classroom provide excellent guidance to help students understand and practice integration by parts and to recognize situations when a particular technique of integration is appropriate. Topic questions for these topics help students to assess their understanding of integration by parts and other integration techniques.

Question BC6

Topic: Maclaurin Series – Alternating Series Error Bound, Radius of Convergence
Max Score: 9
Mean Score: 3.38

What were the responses to this question expected to demonstrate?

In this question, students are told that the Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$. Furthermore, the series converges to $f(x)$ for all x in the interval of convergence and the radius of that interval of convergence is 6.

In part (a) the students are asked to determine whether the Maclaurin series for f converges or diverges at $x = 6$ and to give a reason for their answer. A correct response will determine that at $x = 6,$ the series is $\sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$ and that for all $n \geq 1,$ the terms of this series are larger than the terms of the divergent harmonic series. Therefore, this series diverges by the comparison test.

In part (b) students are told that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2} \cdot \left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Students are asked to show that $|f(-3) - S_3| < \frac{1}{50}$. A correct response will observe that the series for $f(-3)$ is alternating with terms that decrease in magnitude to $0.$ Therefore, by the alternating series error bound, S_3 approximates $f(-3)$ with an error that is no more than the value of the fourth term of the series. The fourth term is $\left|\frac{4+1}{4^2} \cdot \left(-\frac{1}{2}\right)^4\right| = \frac{5}{256},$ which is less than $\frac{1}{50}$.

In part (c) students are asked to find the general term of the Maclaurin series for f' and to find the radius of convergence of the Maclaurin series for $f'.$ A correct response will differentiate the general term of the Maclaurin series for f to find a general term of $\frac{(n+1)x^{n-1}}{n \cdot 6^n}$ and will note that the radius of convergence of the Maclaurin series for f' must be $6,$ because this is the radius of convergence of the Maclaurin series of f.

In part (d) students are given a new Maclaurin series, $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n},$ and asked to use the ratio test to determine the radius of convergence. A correct response will set up a ratio of consecutive terms, $\frac{a_{n+1}}{a_n},$ find the limit of the absolute value of that ratio as $n \to \infty,$ and determine for what values of x the limit is less than 1.

How well did the responses address the course content related to this question? How well did the responses integrate the skill(s) required on this question?

<u>Part (a)</u>

- A majority of the responses successfully identified the general term of the series at $x = 6$.
- A significant number of responses successfully performed a comparison test or a limit comparison test and concluded the series diverges.
- The responses that failed to consider the series at $x = 6$ typically did not make much progress on this part of the problem.

Part (b)

- Most responses identified the fourth term of the Maclaurin series as a bound for the error between $f(-3)$ and S_3.
- A significant number of responses successfully demonstrated that the fourth term of the Maclaurin series was $\frac{5}{256}$ and therefore less than $\frac{1}{50}$.

Part (c)

- Many responses presented the correct general term of the Maclaurin series for f'.
- Some responses wrote the first several terms of the Maclaurin series for f, differentiated each term, and found a pattern for the general term of the series for f'. Many of these responses were successful in finding a correct general term.
- Most responses attempted to use the ratio test to find the interval of convergence for f'.
- Very few responses noted that the radius of convergence for f' must be the same as the radius of convergence for f.

Part (d)

- Most responses presented a correct ratio to use in the ratio test.
- Many responses also correctly found the limit of the presented ratio.
- A significant number of responses provided the correct radius of convergence with supporting work.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses failed to consider the series at $x = 6$.	• At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.
• Some responses drew incorrect conclusions when comparing terms, e.g., $\frac{n+1}{n^2} < \frac{1}{n^2}$.	• For all $n \geq 1$, $\frac{n+1}{n^2} > \frac{1}{n} > \frac{1}{n^2}$.
• Some responses incorrectly applied the ratio test and stated that because $\lim_{n\to\infty} \left\| \frac{n+2}{n+1} \cdot \left(\frac{n}{n+1}\right)^2 \right\| = 1$, the series converges. Other responses concluded that $\lim_{n\to\infty} \left\| \frac{n+1}{n} \right\| = 0$, so the series converges.	• Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \geq 1$, and because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ also diverges.
• Some responses drew an incorrect conclusion from the limit comparison test, deciding that because $\lim_{n\to\infty} \left\| \frac{\frac{n+1}{n^2}}{\frac{1}{n}} \right\| = 1$, the series converges.	• Because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and because $\lim_{n\to\infty} \left\| \frac{\frac{n+1}{n^2}}{\frac{1}{n}} \right\| = 1$, which is finite and

• A significant number of responses incorrectly tried to apply the limit comparison test by comparing series rather than terms of the series. • Many responses struggled with communication, claiming that a term of the series was divergent rather than that the series was divergent.	positive, by the limit comparison test, the series $\sum_{n=1}^{\infty}\frac{n+1}{n^2}$ diverges. • The terms $\frac{n+1}{n^2}$ neither converge nor diverge; the series $\sum_{n=1}^{\infty}\frac{n+1}{n^2}$ diverges.
• In part (b) some responses tried to use the Lagrange Error bound. • A significant number of responses did not observe that the series for $f(-3)$ was alternating, or that the error bound was the alternating series error bound. • Some responses failed to use the fourth term of the series as the error bound. • Some responses stated that the series was a geometric series and tried to find the difference between S_3 and the series sum.	• $\lvert f(-3) - S_3 \rvert <$ alternating series error bound • $f(-3) = \sum_{n=1}^{\infty}\frac{n+1}{n^2}\cdot\left(-\frac{1}{2}\right)^n$ is an alternating series with terms that decrease in magnitude to 0. • By the alternating series error bound, $S_3 = \sum_{n=1}^{3}\frac{n+1}{n^2}$ approximates $f(-3)$ with error at most $\left\lvert \frac{4+1}{4^2}\cdot\left(-\frac{1}{2}\right)^4 \right\rvert = \frac{5}{256} < \frac{1}{50}$.
• In part (c) some responses attempted to differentiate the Maclaurin series for f with respect to n. • Many responses attempted to use the quotient rule to differentiate the general term for f. • Some responses integrated the Maclaurin series for f rather than differentiating it. • Attempts at using the ratio test to find the interval of convergence for f' were generally unsuccessful.	• The derivative with respect to x of $\frac{(n+1)x^n}{n^2 6^n}$ is $\frac{n\cdot(n+1)x^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n\cdot 6^n}$. • $\lim_{n\to\infty}\left\lvert \frac{n(n+2)}{(n+1)^2}\cdot\frac{x}{6} \right\rvert = \frac{\lvert x\rvert}{6}$; $\frac{\lvert x\rvert}{6} < 1 \Rightarrow \lvert x\rvert < 6$. The radius of convergence of the Maclaurin series for f' is 6.
• In part (d) some responses were confused by a general term containing x^{2n} and thus were unable to present a correct ratio. • Many responses did not use absolute values when finding the limit of their ratios, which did not affect scoring in this case but would in most other situations. • Many responses were unable to move from the inequality $x^2 < 3$ to either $-\sqrt{3} < x^2 < \sqrt{3}$ or $\lvert x\rvert < \sqrt{3}$, and so reported a radius of convergence equal to $\frac{1}{3}$, $\frac{1}{9}$, or 3. Some responses presented the inequality $\sqrt{-3} < x^2 < \sqrt{3}$ and claimed the radius of convergence was "$\sqrt{3}$ on the right."	• $\left\lvert \frac{(n+2)\cdot x^{2(n+2)}}{(n+1)^2\cdot 3^{n+1}}\cdot\frac{n^2\cdot 3^n}{(n+1)\cdot x^{2n}} \right\rvert = \left\lvert \frac{(n+2)\cdot x^2}{(n+1)^3\cdot 3} \right\rvert$ • $\lim_{n\to\infty}\left\lvert \frac{(n+2)\cdot x^2}{(n+1)^3\cdot 3} \right\rvert = \left\lvert \frac{x^2}{3} \right\rvert = \frac{x^2}{3}$ • $\frac{x^2}{3} < 1 \Rightarrow x^2 < 3 \Rightarrow -\sqrt{3} < x < \sqrt{3}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve student performance on the exam?

- Students would benefit by practicing applying various convergence tests to all types of series, not only power series.
- Students would benefit by practicing finding the radius of convergence of a given power series. In addition, students need to be careful to provide whichever of the radius of convergence or the interval of convergence is requested.
- Teachers should have students be careful with notation. They should emphasize the differences between terms of a series and the series itself.
- Teachers should not permit students to use shorthand notation that may not be recognized as standard mathematical communication.
- Teachers could emphasize the different error bounds and the situations when each is applicable. In addition, they could emphasize the difference between "error" and "error bound."

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- AP Live Daily Review Videos are excellent resources for helping students to pull together what they need to know about series in preparation for the AP Exam. These resources can be found on AP Classroom under "All Resources" and then "Videos." Review Session 4 from 2022 and Review Session 6 from 2021 are especially relevant to this question.