2024

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free-Response Question 5

☑ Scoring Guidelines

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Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3+4y}$.

Model Solution	Scoring	

(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

$\left.\frac{dy}{dx}\right\|_{(x,\ y)=(2,\ 4)} = \frac{-2(2)}{3+4(4)} = -\frac{4}{19}$	Slope of tangent line	**1 point**
$y \approx 4 - \frac{4}{19}(3-2) = \frac{72}{19}$	Approximation	**1 point**

Scoring notes:

- A response earns the first point for finding $\left.\frac{dy}{dx}\right|_{(x,\ y)=(2,\ 4)} = -\frac{4}{19}$, even if this is not labeled or used as the slope of a tangent line.
- A response that does not explicitly find the value of $\left.\frac{dy}{dx}\right|_{(x,\ y)=(2,\ 4)}$ but uses a slope of $-\frac{4}{19}$ in a linear approximation also earns the first point.
- A response that declares $\left.\frac{dy}{dx}\right|_{(x,\ y)=(2,\ 4)}$ equal to any nonzero value other than $-\frac{4}{19}$ does not earn the first point but is eligible for the second point for a linear approximation at $x = 3$ through the point $(2, 4)$ with a slope of the declared value.
  - The second point cannot be earned with a linear approximation using a slope other than $-\frac{4}{19}$ if that slope has not been declared to be the value of $\left.\frac{dy}{dx}\right|_{(x,\ y)=(2,\ 4)}$.
- The second point cannot be earned for an approximation at any value of x other than 3.
- A response does not have to write the tangent line equation but must clearly demonstrate its use at $x = 3$ in finding the requested approximation in order to earn both points.
- The minimal work required to earn both points is $4 - \frac{4}{19}(3-2)$.

Total for part (a) 2 points

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

$\frac{dy}{dx} = \frac{-2x}{3 + 4y} = 0 \Rightarrow x = 0$	Considers $\frac{dy}{dx} = 0$	**1 point**
And so, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$. However, the point $(0, 1)$ is not on the curve, because $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48.$ Therefore, the horizontal line $y = 1$ is not tangent to the curve.	Answer with reason	**1 point**

Scoring notes:

- The first point can be earned with a response of $\frac{dy}{dx} = 0,$ $-2x = 0,$ or $x = 0,$ OR by identifying the point $(0, 1).$
- To earn the second point a response must provide a reason that the line $y = 1$ is not tangent to the curve. Merely stating "$(0, 1)$ does not lie on the curve" is not sufficient to earn the second point.
- Alternate solution:

  If $y = 1,$ then $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48 \Rightarrow x = \pm\sqrt{43}.$

  $$\left.\frac{dy}{dx}\right|_{(x,\, y)=\left(\pm\sqrt{43},\, 1\right)} = \frac{\pm 2\sqrt{43}}{7} \neq 0$$

  Therefore, the horizontal line $y = 1$ is not tangent to the curve.

  - A response that uses this method earns the first point by using $y = 1$ in $x^2 + 3y + 2y^2 = 48.$
  - A response that fails to consider both $x = +\sqrt{43}$ and $x = -\sqrt{43}$ does not earn the second point.

Total for part (b) 2 points

(c) The curve intersects the positive x-axis at the point $\left(\sqrt{48}, 0\right)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

At the point $\left(\sqrt{48}, 0\right)$, the slope of the line tangent to the curve is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3 + 4(0)}$. The denominator of $\frac{dy}{dx}$ is $3 + 4(0)$, which does not equal 0. Therefore, the line tangent to the curve at this point is not vertical.	Answer with reason	**1 point**

Scoring notes:

- A response does not need to consider the numerator of $\left.\frac{dy}{dx}\right|_{(x,\, y)=\left(\sqrt{48},\, 0\right)}$ in order to earn this point; considering the denominator is sufficient.
- To earn this point a response must clearly demonstrate that the slope of the tangent line at the point $\left(\sqrt{48}, 0\right)$ is defined and answer "no."
  - Such demonstrations include, but are not limited to, the following:
    - $3 + 4(0) \neq 0$
    - At $\left(\sqrt{48}, 0\right)$, $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$.
    - $\frac{-2\sqrt{48}}{3}$
    - When $3 + 4y = 0$, $y \neq 0$.

Total for part (c) 1 point

(d) For time $t \geq 0,$ a particle is moving along another curve defined by the equation $y^3 + 2xy = 24.$ At the instant the particle is at the point $(4, 2),$ the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

$3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$	Attempts implicit differentiation	**1 point**
	$3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$	**1 point**
$\frac{dy}{dt} = -2$	Uses $\frac{dy}{dt} = \pm 2$	**1 point**
$3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4} = 10$ The rate of change with respect to time in the x-coordinate is 10 units per second.	Answer	**1 point**

Scoring notes:

- The first point is earned for implicitly differentiating $y^3 + 2xy = 24$ with respect to t with at most one error.
    - o The first point can also be earned by correctly differentiating with respect to x: $3y^2 \frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0.$
- The second point is earned for an equation equivalent to $3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0.$
- A response does not need to explicitly declare $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in order to earn the third point; this point may be earned by correctly substituting $\frac{dy}{dt} = -2$ or $\frac{dy}{dt} = 2$ in the implicitly differentiated equation. However, a response that uses both $\frac{dy}{dt} = -2$ and $\frac{dy}{dt} = 2$ in the implicitly differentiated equation does not earn the third point.
- The fourth point cannot be earned without the first 3 points. The fourth point is earned only for the value of 10 with no mistakes in supporting work.
    - o Note that a response that uses $\frac{dy}{dt} = 2$ and then mishandles subtracting 40 from both sides of the equation, e.g., $3(2)^2(2) + 2(4)(2) + 2(2)\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{40}{4},$ does not earn the fourth point.

Total for part (d) 4 points

Total for question 5 9 points

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$\frac{dy}{dx}\Big|_{(2,4)} = \frac{-2(2)}{3+4(4)} = \frac{-4}{3+16} = \frac{-4}{19}$

tangent line at (2,4): $y = -\frac{4}{19}(x-2)+4$

$y = -\frac{4}{19}(3-2)+4$

$= -\frac{4}{19}(1)+4$

$= -\frac{4}{19}+4$

Response for question 5(b)

If y=1 is tangent to the curve, at some point $\frac{dy}{dx}$ should be 0 when y=1.

$0 = \frac{-2x}{3+4y}$

$0 = \frac{-2x}{3+4}$

$0 = -\frac{2}{7}x$

$x=0$

$0^2+3(1)+2(1)^2=48$

$3+2 \neq 48$

y=1 is NOT tangent to the curve since the calculated value of x when y=1 and $\frac{dy}{dx}=0$ (where y=1 would intersect the graph) is 0, while (0,1) does not fall on the curve.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$$\left.\frac{dy}{dx}\right|_{(\sqrt{48},0)} = \frac{-2(\sqrt{48})}{3+4(0)} = \frac{-2\sqrt{48}}{3}$$

The line tangent to the curve at the point $(\sqrt{48}, 0)$ is not vertical because solving for $\frac{dy}{dx}$ at this point gives a defined slope (not an undefined one that would be expected of the presence of a vertical tangent line).

Response for question 5(d)

$$\frac{dy}{dt} = -2 \ \frac{\text{units}}{\text{second}}$$

$$\frac{d}{dt}(y^3 + 2xy) = (24)\frac{d}{dt}$$

$$3y^2\left(\frac{dy}{dt}\right) + 2\left(\frac{dx}{dt}\right)(y) + 2x\left(\frac{dy}{dt}\right) = 0$$

$$3(2^2)(-2) + 2\left(\frac{dx}{dt}\right)(2) + 2(4)(-2) = 0$$

$$-24 + 4\frac{dx}{dt} - 16 = 0$$

$$4\frac{dx}{dt} = 40$$

$$\frac{dx}{dt} = 10 \ \frac{\text{units}}{\text{second}}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$$\frac{dy}{dx} = \frac{-2(2)}{3+4(4)} = \frac{-4}{19}$$

$$y-4 = \frac{-4}{19}(x-2)$$

$19 \times 4 = 76$

$$y-4 = -\frac{4}{19}x + \frac{8}{19}$$

$$y = -\frac{4}{19}x + \frac{76}{19}$$

$$\boxed{y(3) = -\frac{4}{19}(3) + \frac{76}{19}}$$

Response for question 5(b)

$$\frac{dy}{dx} = \frac{-2(\sqrt{43})}{3+4} = \frac{-2\sqrt{43}}{7}$$

$$x^2 + 3(1) + 2(1)^2 = 48$$

$$x^2 + 5 = 48$$

$$x^2 = 43$$

$$x = \sqrt{43}$$

No at the point where $y=1$, $x=\sqrt{43}$ at this point the tangent line is not $y=1$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5223/12

5 5 5 5 5 **NO CALCULATOR ALLOWED** 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$\frac{dy}{dx} = \frac{-2(\sqrt{48})}{3m\cos} = \frac{-2\sqrt{48}}{3} = -\frac{8\sqrt{3}}{3}$

$y - 0 = -\frac{8\sqrt{3}}{3}(x - 4\sqrt{3})$

At this point the line tangent is not vertical since it has a slope.

Response for question 5(d)

$y^3 + 2xy = 24$

$3y^2 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$

$3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0$

$3(4)(-2) + (-16) + 4\frac{dx}{dt} = 0$

$-24 + 16 + 4\frac{dx}{dt} = 0$

$-8 + 4\frac{dx}{dt} = 0$

$4\frac{dx}{dt} = 8$

$\frac{dx}{dt} = 2$ units per second

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5223/13

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$L(x) = f(2) + f'(2)(x-2)$

$= f(2) + f'(2)(3-2)$

$= 4 - \frac{7}{4}(1)$

$= 4 - \frac{7}{4}$

$= \frac{9}{4}$

$3+4y = -4$

$4y = -7$

Response for question 5(b)

$\frac{dy}{dx} = \frac{-2x}{3+4y}$

$(3+4y)\frac{dy}{dx} = -2x$

$(3+4(1))\frac{dy}{dx} = -2(\sqrt{43})$

$7\frac{dy}{dx} = -2\sqrt{43}$

$\frac{dy}{dx} = \frac{-2\sqrt{43}}{7}$

$x^2+3(1)+2(1)^2=48$

$x^2+3+2=48$

$x^2=43$

$x=\sqrt{43}$

No, the horizontal line y=1 has a slope of 0 while at $(\sqrt{43},1)$, the slope is $\frac{-2\sqrt{43}}{7}$. The slope of the function and of y=-1 are not the same.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5223/12

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$(\sqrt{48}, 0)$ $\frac{dy}{dx} = \frac{-2x}{3+4y}$

$\frac{dy}{dx} = \frac{-2\sqrt{48}}{3+4(0)}$

$\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$

No, the slope of the curve at $(\sqrt{48}, 0)$ is not undefined. therefore, the line tangent to the curve at this point is not vertical.

Response for question 5(d)

$y^3 + 2xy = 24$

$3y^2 \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y = 0$

$\frac{dy}{dx}(3y^2 + 2x) = -2y$

$\frac{dy}{dx} = \frac{-2y}{(3y^2+2x)}$

$\frac{2}{dx} = \frac{-2(2)}{(3(2)^2+2(4))}$

$\frac{2}{dx} = \frac{-4}{20}$

$\frac{2}{dx} = \frac{-1}{5}$

$-10 = dx$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5223/13

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This question began by asking students to consider the curve implicitly defined by the equation $x^2 + 3y + 2y^2 = 48$. Students were given that for this curve, $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$.

In part (a) students were asked to use the line tangent to the curve at the point $(2, 4)$ to approximate the y-coordinate of the point on the curve with x-coordinate 3. A correct response will use the given equation for $\frac{dy}{dx}$ to find the slope of this tangent line and then use the equation of the tangent line at $(2, 4)$, $y = 4 - \frac{4}{19}(x - 2)$, to approximate the y-coordinate when $x = 3$.

In part (b) students were asked to determine whether the horizontal line $y = 1$ is tangent to the implicitly defined curve. A correct response will recognize that a horizontal tangent line must have a slope of 0, and therefore must satisfy $\frac{dy}{dx} = 0$. Therefore, a the horizontal line $y = 1$ must pass through the point $(0, 1)$. The response will then find that $0^2 + 3 \cdot 1 + 2 \cdot 1^2 \neq 48$, so the line $y = 1$ cannot be tangent to the given curve.

Alternatively, a correct response could use $y = 1$ and the equation of the curve to find $x^2 + 3 \cdot 1 + 2 \cdot 1^2 = 48$, which happens only when $x = \pm\sqrt{43}$. However, at either point $\left(\pm\sqrt{43}, 1\right)$, the slope $\frac{dy}{dx} = \frac{-2x}{3 + 4y}$ is not 0, so the line cannot be tangent to the given curve.

In part (c) students were told that the curve intersects the x-axis at the point $\left(\sqrt{48}, 0\right)$ and asked whether the line tangent to the curve at this point is vertical. A correct response will find that at this point on the curve the slope of the tangent line is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3 + 4 \cdot 0}$, with denominator $3 + 4 \cdot 0 \neq 0$. Thus, the line tangent to the curve at the point $\left(\sqrt{48}, 0\right)$ is not vertical.

In part (d) students were given a different equation, $y^3 + 2xy = 24$, and were told that on the curve implicitly defined by this equation, at the instant when the particle is at the point $(4, 2)$, the y-coordinate of the particle's position was decreasing at a rate of 2 units per second. Students were asked to find the rate of change of the x-coordinate of the particle's position with respect to time at that instant. A correct response will implicitly differentiate the equation with respect to t, use $x = 4$, $y = 2$, and $\frac{dy}{dt} = -2$ in the resulting differentiated equation, and solve for $\frac{dx}{dt}$.

Question 5 (continued)

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 4 points in part (d).

In part (a) the response would have earned the first point in line 1 with $\frac{-2(2)}{3+4(4)}$ without simplification. In this case, correct simplification of $\frac{-4}{19}$ at the end of line 1 earned the first point. The response would have earned the second point with $y = -\frac{4}{19}(3-2)+4$ in line 3. In this case, correct simplification of $-\frac{4}{19}+4$ in line 5 earned the second point.

In part (b) the response earned the first point in line 2 on the left with $0 = \frac{-2x}{3+4y}$. The response earned the second point with a correct answer in lines 4 through 7 on the right supported by numerical reasoning in lines 2 and 3 on the right demonstrating that $(0, 1)$ does not fall on the curve.

In part (c) the response would have earned the point with $\frac{-2(\sqrt{48})}{3+4(0)}$ in line 1 and "not vertical" in line 3. The response earned the point because $\frac{-2\sqrt{48}}{3}$ is simplified correctly and the additional reasoning in lines 2 through 6 is without error.

In part (d) the response earned the first and second points in line 2 with $3y^2\left(\frac{dy}{dt}\right)+2\left(\frac{dx}{dt}\right)(y)+2x\left(\frac{dy}{dt}\right)=0.$ Note that the response is not penalized for incorrect placement of the $\frac{d}{dt}$ operator on the right side of the equation in line 1. The response earned the third point by using -2 for $\frac{dy}{dt}$ in line 3. The response earned the fourth point with the correct answer in line 6.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d).

In part (a) the response would have earned the first point with $\frac{-2(2)}{3+4(4)}$ in the middle of line 1 on the right. The response earned the first point with a correctly simplified value of $\frac{-4}{19}$ at the end of the same line. The response did not earn the second point because the boxed answer of "$y(3) = \frac{-4}{19}(3) + \frac{76}{19}$" is incorrect.

In part (b) the response earned the first point by using 1 for y in line 1 on the left. The response did not earn the second point because it did not consider both $\pm\sqrt{43}$.

Question 5 (continued)

In part (c) the response earned the point with a calculated $\frac{dy}{dx}$ of $\frac{-8\sqrt{3}}{3}$ in line 1 followed by the reason of "At this point the line tangent is not vertical since it has a slope." Note that the equation of the tangent line is given but not necessary.

In part (d) the response earned the first and second points with $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$ in line 2. The response earned the third point by using -2 for $\frac{dy}{dt}$ in line 3. The response did not earn the fourth point because the boxed answer of $\frac{dx}{dt} = 2$ is incorrect.

Sample: 5C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d).

In part (a) the response did not earn the first point because no value equivalent to $\frac{-4}{19}$ was found. The response did not earn the second point because the value of $-\frac{7}{4}$ that is being used in the third line was never declared to be $\frac{dy}{dx}$ before appearing in the linear approximation $4 - \frac{7}{4}(1)$.

In part (b) the response earned the first point by using 1 for y in line 1 on the right. The response did not earn the second point because it does not consider the slope for $x = -\sqrt{43}$.

In part (c) the response would have earned the point with $\frac{-2\sqrt{48}}{3 + 4(0)}$ in line 2 and "No" in line 4. The response earned the point with a correct simplification of $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3}$ in line 3 and reasoning in lines 4 through 6 that is without error.

In part (d) the response earned the first point with a correct derivative with respect to x, $3y^2\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$ in the second line. The response did not earn the second point because the derivative was not taken with respect to t. The response did not earn the third point because it did not use 2 for $\frac{dy}{dt}$. The response is not eligible for the fourth point because it did not earn the first 3 points.