# Chief Reader Report on Student Responses:
## 2022 AP® Calculus AB/Calculus BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC): 1,166

**Calculus AB**

- Number of Students Scored: 268,352
- Score Distribution

| Exam Score | N | %At |
|---|---|---|
| 5 | 54,862 | 20.4 |
| 4 | 43,306 | 16.1 |
| 3 | 51,206 | 19.1 |
| 2 | 60,655 | 22.6 |
| 1 | 58,323 | 21.7 |

- Global Mean: 2.91

**Calculus BC**

- Number of Students Scored: 120,238
- Score Distribution

| Exam Score | N | %At |
|---|---|---|
| 5 | 49,544 | 41.2 |
| 4 | 18,768 | 15.6 |
| 3 | 24,115 | 20.1 |
| 2 | 19,668 | 16.4 |
| 1 | 8,143 | 6.8 |

- Global Mean: 3.68

**Calculus BC Calculus AB Subscore**

- Number of Students Scored: 120,276
- Score Distribution

| Exam Score | N | %At |
|---|---|---|
| 5 | 58,307 | 48.5 |
| 4 | 25,116 | 20.9 |
| 3 | 14,142 | 11.8 |
| 2 | 14,616 | 12.2 |
| 1 | 8,095 | 6.7 |

- Global Mean: 3.92

* The number of students with Calculus AB subscores may differ slightly from the number of students who took the AP Calculus BC Exam due to exam administration incidents.

The following comments on the 2022 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Julie Clark of Hollins University. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

## Question AB1/BC1

**Topic:** Modeling Rates
**Max Score:** 9
**Mean Score: AB1** 3.22
**Mean Score: BC1** 4.79

### *What were the responses to this question expected to demonstrate?*

The context of this problem is vehicles arriving at a toll plaza at a rate of $A(t) = 450\sqrt{\sin(0.62t)}$ vehicles per hour, with time $t$ measured in hours after 5 A.M., when there are no vehicles in line.

In part (a) students were asked to write an integral expression that gives the total number of vehicles that arrive at the plaza from time $t = 1$ to time $t = 5.$ A correct response would report $\int_1^5 A(t)\, dt.$

In part (b) students were asked to find the average value of the rate of vehicles arriving at the toll plaza over the same time interval, $t = 1$ to $t = 5.$ A correct response would report $\frac{1}{4}\int_1^5 A(t)\, dt$ and then evaluate this definite integral using a calculator to find an average value of $375.537.$ (The units, vehicles per hour, were given in the statement of the problem.)

In part (c) students were asked to reason whether the rate of vehicles arriving at the toll plaza is increasing or decreasing at 6 A.M., when $t = 1.$ A correct response would use a calculator to determine that $A'(1),$ the derivative of the function $A(t)$ at this time, is positive ( $A'(1) = 148.947$ ) and conclude that because $A'(1)$ is positive, the rate of vehicles arriving at the toll plaza is increasing.

Finally, in part (d) students were told that a line of vehicles forms when $A(t) \geq 400$ and the number of vehicles in line is given by the function $N(t) = \int_a^t (A(x) - 400)\, dx,$ where $a$ denotes the time, $a \leq t \leq 4,$ when the line first begins to form. Students were asked to find the greatest number of vehicles in line at the plaza, to the nearest whole number, in the time interval $a \leq t \leq 4$ and to justify their answer. A correct response would recognize that the greatest number of vehicles is the maximum value of $N(t)$ on the closed interval $a \leq t \leq 4.$ To find this maximum, a response should first determine the times $t,$ $0 < t \leq 4,$ when the derivative of $N(t)$ is $0.$ This requires using the Fundamental Theorem of Calculus to find $N'(t) = A(t) - 400$ and then using a calculator to determine that $N'(t)$ is equal to zero when $t = a = 1.469372$ and when $t = b = 3.597713.$ A response should then evaluate the function $N(t)$ at each of the values $t = a,$ $t = b,$ and $t = 4$ to determine that the greatest number of vehicles in line is $N(b) = 71.$

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses used the content knowledge that integrating a rate function provides the accumulation over an interval. Most responses also demonstrated notational fluency by correctly displaying the required definite integral, although occasionally, the responses failed to include the differential, $dt.$

In part (b) a good number of responses demonstrated the content knowledge of the average value of a function, presented the correct integral expression setup, and were able to use their calculators correctly to provide the correct numerical value.

In part (c) a majority of the responses were successful in determining the sign of the derivative of a given function evaluated at a specific point and using that sign to determine whether the function was increasing or decreasing at that specific point.

In part (d) responses that recognized the need to start by setting the derivative of the given function, $N(t),$ equal to $0$ (frequently by reporting the equivalent statement $A(t) = 400$ ) performed well on this part of the problem. Such responses were able to locate the necessary critical values of the function $N$, evaluate $N$ at both critical values and the interval endpoints, and identify the location of the relative maximum of $N$.

### *What common student misconceptions or gaps in knowledge were seen in the responses to this question?*

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) responses sometimes struggled with notational fluency by writing incorrect statements, such as $A(t) = \int_1^5 A(t)\,dt$, $\sum \int_1^5 A(t)\,dt$, or $\int_1^5 A(t)\,dt + C$.<br>• Some responses reversed the limits of integration or used incorrect limits of $6$ and $10$.<br>• Some responses presented $A'(t)$ as the integrand or incorrectly copied the given expression for $A(t)$ as the integrand. | • Total number of vehicles $= \int_1^5 A(t)\,dt$<br>OR<br>Total number of vehicles $= \int_1^5 450\sqrt{\sin(0.62)}\,dt$ |
| • In part (b) several responses reported the average rate of change of $A$, $\frac{A(5)-A(1)}{4}$, or $\frac{1}{5-1}\int_1^5 A'(t)\,dt$.<br>• Poor communication, such as $\int_1^5 A'(t)\,dt = \frac{1}{4}\int_1^5 A'(t)\,dt$ or the equivalent, was frequent.<br>• Quite a few responses displayed arithmetic errors, such as $\frac{1}{5-1} = \frac{1}{5}$.<br>• Some responses rounded the numerical answer in this part to a whole number. | • $\frac{1}{5-1}\int_1^5 A(t)\,dt = 375.537$<br>• $\int_1^5 A(t)\,dt = 1502.147865$<br>Average rate $= \frac{1502.147865}{4}$<br>• $\frac{1}{5-1}\int_1^5 A(t)\,dt = \frac{1}{4}\int_1^5 A(t)\,dt = 375.537$ |
| • In part (c) poor communication was frequent, including using incorrect notation, such as $\frac{dA(1)}{dt}$, or reporting "the rate of $A(t)$" rather than "the rate of change of $A(t)$."<br>• Many responses referenced only $A'(t)$ rather than $A'(1)$.<br>• Responses frequently presented errors in attempts to symbolically differentiate $A(t)$. | • $A'(1) > 0$, therefore, the rate at which vehicles arrive is increasing.<br>• When $t = 1$, $A'(t) > 0$, so the rate at which vehicles arrive at the plaza is increasing.<br>• $A'(t) = \frac{450}{2}\cdot\frac{0.62\cos(0.62t)}{\sqrt{\sin(0.62t)}}$ |

| | |
|---|---|
| • In part (d) some responses never specifically reported the equation solved using a calculator, either $N'(t) = 0$ or $A(t) = 400$.<br><br>• Several responses failed to find the interior critical point ($t = b = 3.597713$) and, therefore, could not complete the Candidates Test or determine the location or value of the maximum.<br><br>• Many responses did not complete the Candidates Test by evaluating $N(t)$ at both endpoints and the interior critical point. Some justified a local maximum rather than a global maximum on the interval $[a, 4]$. | • $N'(t) = 0 \Rightarrow A(t) = 400$<br>$\Rightarrow t = a = 1.469372$ or $t = b = 3.597713$.<br><br>• $N(a) = 0,\ N(b) = 71.254,\ N(4) = 62.338.$ Therefore, the greatest number of vehicles in line was 71.<br><br>• The only critical point in $[a, 4]$ is $t = b = 3.597713$, and $N'(t)$ changes from positive to negative at $t = b$. Therefore, the greatest number of vehicles in line was $N(b) = 71$. |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Teachers should work to help students communicate concisely and correctly. Often one well-worded sentence (e.g., $A'(1) > 0$; therefore, the rate of vehicles arriving is increasing) will provide an explanation, reason, or justification that is better than wordy statements that may include irrelevant or incorrect information.
- Teachers can provide opportunities for practicing notational fluency by providing multiple representations of communicated mathematics. For example, $A'(1) = \left.\frac{dA}{dt}\right|_{t=1}$.
- Teachers could provide students with practice distinguishing between the average rate of change of a function and the average value of a function.
- Teachers should help students to understand that a "rate of change" cannot always be thought of as a velocity and that defining a function $A(t)$ as a rate of change means that $A(t)$ is a derivative of some function.
- Teachers should remind students to always make sure their calculators are in radian mode.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- An important concept assessed in part (a) of AB1/BC1 was interpreting an accumulation problem as a definite integral (LO CHA-4.D). To set up the correct integral requires understanding that $A(t) = 450\sqrt{\sin(0.62t)}$ gives a rate of change in cars per hour and that the net change in the number of cars over the time interval from time $t = 1$ to time $t = 5$ is given by $\int_1^5 A(t)\,dt$. Incorrectly presenting an integrand of $A'(t)$, for example, may indicate a general understanding that a rate should be integrated while misunderstanding that $A(t)$ is the rate required in this context (see Topic 8.3 in the AP Calculus AB and BC Course and Exam Description [CED], page 151).
    - Video 1 in Topic 8.3 on AP Classroom develops the abstract calculus concepts required to set up an appropriate response to this question.
    - Video 2 in Topic 8.3 on AP Classroom introduces application of these concepts in context, including consideration of appropriate rounding. The first example presented in the video starts with a rate, $V'$. The second example in the video features a rate, $r$. It is important to emphasize that in both examples, we are integrating a rate.

- To respond successfully in part (d), a student must recognize that the question is asking for the absolute maximum value for the function $N(t) = \int_a^t (A(x) - 400)\,dx$, identify critical points using a calculator, and apply the Candidates Test to determine the absolute minimum value for the number of vehicles in line on the given interval. Responses that began by setting $A(t) = 400$ (or equivalent) tended to be successful with the rest of the question, suggesting that not recognizing this as an optimization problem or not knowing how to differentiate $N(t)$ were potential barriers to success.
  - Using the instructional strategy "Marking the Text" (page 208 of the CED) is a good way to teach students how to identify the question being asked ("greatest number of vehicles"), along with other important information in the text (skills 1.A and 2.B)
  - It is essential that students develop the understanding of the Fundamental Theorem of Calculus needed to differentiate $N(t)$ (Topic 6.4, CED). Video 1 in Topic 6.4 on AP Classroom provides a clear explanation of how to find $N'(t)$.
- Across content, developing strong communication and notation skills (Mathematical Practice 4) is important. Requiring students to clearly communicate their setups, work, and mathematical reasoning is essential both to surface conceptual misunderstandings and to develop mastery of these skills. The Instructional Focus Section of the CED includes strategies specific to Mathematical Practice 4, starting on page 219. Strategies for teaching skill 4.A (Use precise mathematical language), skill 4.C (Use appropriate mathematical symbols and notation), and skill 4.E (Apply appropriate rounding procedures) would be particularly helpful to developing the mastery needed for students to excel on questions similar to AB1/BC1.

## Question AB2

**Topic:** Area-Volume with Related Rates
**Max Score:** 9
**Mean Score:** 3.34

### *What were the responses to this question expected to demonstrate?*

In this problem students were provided graphs of the functions $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$ and told that the graphs intersect at $x = -2$ and $x = B$, where $B > 0$.

In part (a) students were asked to find the area of the region enclosed by the graphs of $f$ and $g$. A correct response provides the setup of the definite integral of $f(x) - g(x)$ from $x = -2$ to $x = B$. The response must determine the value of $B$ (although this value need not be presented) and then use this value to evaluate the integral and find an area of $3.604$.

In part (b) the function $h(x)$ is defined to be the vertical distance between the graphs of $f$ and $g$, and students were asked to reason whether $h$ is increasing or decreasing at $x = -0.5$. A correct response would recognize that the vertical distance between the graphs of $f$ and $g$ is $f(x) - g(x)$ and then evaluate the derivative $h'(x) = f'(x) - g'(x)$ at $x = -0.5$. Because this value is negative, the response should conclude that $h$ is decreasing when $x = -0.5$.

In part (c) students were told that the region enclosed by the graphs of $f$ and $g$ is the base of a solid with cross sections of the solid taken perpendicular to the $x$-axis that are squares. Students were asked to find the volume of the solid. A correct response would realize that the area of a cross section is $(f(x) - g(x))^2$ and would find the requested volume by integrating this area from $x = -2$ to $x = B$.

In part (d) students were told that a vertical line in the $xy$-plane travels from left to right along the base of the solid described in part (c) at a constant rate of $7$ units per second. Students were asked to find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$. A correct response would again use the area function from part (c), $A(x) = (f(x) - g(x))^2$ and the chain rule to find $\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$. The response should then use a calculator to find $A'(-0.5)$ and multiply this value by the given value of $\frac{dx}{dt} = 7$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses showed familiarity with the computation of the area between the curves. Only a few responses reversed the integrand, writing $g(x) - f(x)$ instead of $f(x) - g(x)$. A few responses unnecessarily divided the area into multiple pieces and used definite integrals to find each area separately. Almost all responses that presented a correct definite integral evaluated the integral correctly (using a calculator). Many responses did calculate or present a correct value for the upper limit of integration, $B$.

In part (b) most responses interpreted the vertical distance correctly as $f(x) - g(x)$, although frequently they did not denote this distance as $h(x)$ in their explanations. Responses that considered the value of $h'(-0.5)$ directly using their calculators or considered $f'(-0.5) - g'(-0.5)$ were generally successful in explaining that the vertical distance was decreasing when $x = -0.5$. Responses that attempted to compare $f'(x)$ and $g'(x)$ verbally sometimes made errors in their explanations by not referencing $x = -0.5$ or by not clearly linking their explanation with calculus concepts.

In part (c) a majority of the responses demonstrated an understanding of how to find volumes of solids with given cross sections. Almost all responses that presented a correct definite integral evaluated it correctly, but as in part (a), errors in attempting to simplify an analytic presentation of the integrand were quite common (usually a failure to distribute the subtraction across parentheses). There were not many responses in part (d) that demonstrated an ability to display the cross-sectional area as a (correct) function of $x$ or interpret the given rate as $\frac{dx}{dt}$. In addition, it was rare for a response to demonstrate an understanding that the cross-sectional area was also a function of $t$ and, therefore, the chain rule was needed in order to compute $\frac{dA}{dt}$.

### *What common student misconceptions or gaps in knowledge were seen in the responses to this question?*

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) responses too frequently presented an analytic expression for $f(x) - g(x)$ but simplified incorrectly by failing to distribute the $-1$. For example, $\ln(x+3) - x^4 + 2x^3$. | • Store the functions $f(x)$ and $g(x)$ in the calculator, then use the calculator to find $\int_{-2}^{B} (f(x) - g(x))\, dx = 3.604$. |
| • In part (b) responses made errors in computing and/or simplifying analytic expressions for $h'(x)$. | • Use a calculator with the stored functions $f(x)$ and $g(x)$ to find $h'(-0.5) = f'(-0.5) - g'(-0.5)$. |
| • In part (c) some responses used integrands of $(f(x))^2 - (g(x))^2$ or $\pi(f(x) - g(x))^2$.<br>• Errors in copying, expanding, or simplifying analytic expressions for $A(x) = (f(x) - g(x))^2$ were quite common. | • $\int_{-2}^{B} (f(x) - g(x))^2\, dx = 5.340$ |
| • In part (d) many responses created a variable $s$ in order to provide an equation for the area, $A = s^2$, then found $\frac{dA}{dt} = 2s\frac{ds}{dt}$ but failed to realize that $s = f(x) - g(x)$. | • Because $s = f(x) - g(x)$, $A(x) = (f(x) - g(x))^2$<br>$\Rightarrow \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$<br>$\Rightarrow \left.\frac{dA}{dt}\right\rvert_{x=-0.5} = A'(-0.5) \cdot 7$ |

### ***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Communication of mathematical results continues to be a problem for many students. This was particularly true in part (b), where students who used $h(x)$ in their responses tended to produce correct and briefer answers more often. Students who tried to explain their correct answer verbally often entangled themselves in incorrect or nonmathematical terms, such as "the rate of $g$ is increasing" or "increases more steeply." Such terms are easily misinterpreted. Teachers should model using a standard list of mathematical terms to describe the behavior and properties of functions, avoiding colloquial terms as much as possible.

- Teachers could provide more practice using graphing calculators to store functions in order to easily calculate square roots, derivatives, integrals, and sums of squares. Students should be encouraged to use given function names when presenting the expression used in the graphing calculator rather than trying to rewrite the entire function definition, as this often results in a "copy error." Because the calculator can find the numerical value of a derivative at a point, that capability should be used. Teachers should provide numerous situations in which using a calculator is absolutely necessary and should emphasize the most appropriate ways to use the calculator in addressing these situations.

- Modeling quantities using functions is a fundamental activity in precalculus and calculus. In part (d) most students were unable to model the cross-sectional area as a function of time because they could not express the cross-sectional area as a function of the $x$-coordinate of the cross section or to understand that the $x$-coordinate was a function of time. Teachers should provide opportunities for students to use composition of functions to model quantities as functions of time as they discuss the chain rule and related rates problems.

- Mathematical notation is an ongoing problem. Particularly in part (d), many students introduced variables without clearly defining them. This made it difficult for them to recognize the connection between their variables and the functions given in the question. Teachers should model defining variables whenever they are first used in a solution.

### ***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- As noted in the online resource *AP Calculus: Use of Graphing Calculators*, (linked here and in the CED), students need frequent opportunities to practice using their calculators so that they may become adept at their use. This resource identifies the four graphing calculator capabilities students are expected to be able to use in calculator-active questions and advises students of the importance of showing the setup for work performed in a calculator, along with the answer.

  In part (a), for example, as in all calculator-active questions, a response must include the setup for a calculation performed in a calculator—in this case, the appropriate definite integral, along with the answer. Because part (a) is worth three points, presenting an unsupported answer would be a costly mistake.

- The introductions to each unit in the CED are resources for developing conceptual understanding and mastery of the mathematical practices, as well as preparing for the exam. For example, the "Preparing for the AP Exam" section of the introduction to Unit 6 of the CED (page 111) addresses issues associated with calculator usage and communication relevant to AB2, including the need to present setups for calculations and to be careful about parentheses usage and other details of clear communication.

  A complete response to AB2 would need to present $\int_{-2}^{B}(f(x)-g(x))\,dx = 3.604$ in part (a), $f'(-0.5)-g'(-0.5)$ (or equivalent) in part (b), $\int_{-2}^{B}(f(x)-g(x))^2\,dx = 5.340$ in part (c), and the expression $A(x)=(f(x)-g(x))^2$ in part (d). Some students exposed themselves to repeated parentheses errors by substituting expressions for $f(x)$ and $g(x)$ into one or more of these expressions, as in $\ln(x+3)-x^4+2x^3$. One way to be careful about parentheses usage (and copy errors) in a calculator-active question is to simply refer to the functions using the names given to you, rather than by their full analytical expressions.

- AP Daily Videos provided on AP Classroom are very helpful resources for teaching and learning AP Calculus:

- Part (a): Video 2 for Topic 8.4 on AP Classroom illustrates how to find the area of a region bounded by the graphs of two functions, including an example of the work needed in a calculator-active question. Although not featured in this video, we recommend storing intermediate values, such as $B = 0.781975,$ to avoid errors introduced by premature rounding of intermediate values.

- Part (b): Video 1 for Topic 4.3 on AP Classroom provides several examples of how to determine whether a quantity, such as $h(x),$ is increasing or decreasing based on the sign of its derivative.

- Part (c) may be conceptually difficult for some students. The three videos provided with Topic 8.7 on AP Classroom start with an introduction to finding volumes of solids with square cross-sections and build to more complex versions of the question.

- In part (d) some students had difficulty with writing the expression for area needed to set up the related rates question. These students did not make the connection between the questions in parts (c) and (d). Others did not use the chain rule in the differentiation step. Excellent videos on solving related rates problems can be found with Topics 4.4 and 4.5 on AP Classroom. Video 3 for Topic 3.2 on AP Classroom provides examples of how to correctly handle implicit differentiation on AP-style questions, including the use of the chain rule and other differentiation rules.

# Question AB3/BC3

**Topic:** Graphical Analysis of Functions
**Max Score:** 9
**Mean Score: AB3** 2.36
**Mean Score: BC3** 4.25

### *What were the responses to this question expected to demonstrate?*

In this problem the graph of a function $f'$, which consists of a semicircle and two line segments on the interval $0 \le x \le 7$, is provided. It is also given that this is the graph of the derivative of a differentiable function $f$ with $f(4) = 3$.

In part (a) students were asked to find $f(0)$ and $f(5)$. To find $f(0)$ a correct response uses geometry and the Fundamental Theorem of Calculus to calculate the signed area of the semicircle, $\int_0^4 f'(x)\,dx = -2\pi$, and subtracts this value from the initial condition, $f(4) = 3$, to obtain a value of $3 + 2\pi$. To find $f(5)$ a correct response would add the initial condition to the signed area $\int_4^5 f'(x)\,dx = \frac{1}{2}$, found using geometry, to obtain a value of $\frac{7}{2}$.

In part (b) students were asked to find the $x$-coordinates of all points of inflection on the graph of $f$ for $0 < x < 7$ and to justify their answers. A correct response would use the given graph to determine that the graph of $f'(x)$ changes from decreasing to increasing, or vice versa, at the points $x = 2$ and $x = 6$. Therefore, these are the inflection points of the graph of $f$.

In part (c) students were told that $g(x) = f(x) - x$ and are asked to determine on which intervals, if any, the function $g$ is decreasing. A correct response would find that $g'(x) = f'(x) - 1$ and then use the given graph of $f'$ to determine that when $0 \le x \le 5$, $f'(x) \le 1 \Rightarrow g'(x) \le 0$. Therefore, $g$ is decreasing on the interval $0 \le x \le 5$.

In part (d) students were asked to find the absolute minimum value of $g(x) = f(x) - x$ on the interval $0 \le x \le 7$. A correct response would use the work from part (c) to conclude $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$. Thus the absolute minimum of $g$ occurs at $x = 5$. Using the work from part (a), which found the value of $f(5)$, the absolute minimum value of $g$ is $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses recognized the need to calculate areas under the given curve in order to find the requested function values, and most were able to use geometry to calculate the areas of the relevant regions. However, many responses were not sure how to incorporate the initial condition, $f(4) = 3$, into their calculations.

In part (b) a majority of the responses successfully identified the inflection point at $x = 2$; they were not as successful recognizing the additional inflection point at $x = 6$. Responses that centered their justifications around the behavior of the graph of $f'$ were usually successful.

In part (c) almost all responses were successful in computing $g'(x) = f'(x) - 1$, and many also reported that $g$ is decreasing on $0 < x < 5$, although they rarely provided a clear explanation of how they found this interval.

In part (d) many responses successfully applied the Candidates Test and found the absolute minimum value of $g$ to be $-\frac{3}{2}$. Responses that discussed the decreasing-to-increasing behavior of $g$ around $x = 5$ often provided only a local argument that did not appeal to the full interval $0 \le x \le 7$ and, therefore, did not successfully justify their answer.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) the most common error was incorrectly reporting $f(0) = 3 - 2\pi$. | • $f(0) = f(4) - \int_0^4 f'(x)\,dx = 3 - (-2\pi) = 3 + 2\pi$ |
| • In part (b) many responses failed to list $x = 6$ as an inflection point, presumably because the graph of $f'$ is not differentiable at $x = 6$.<br>• Many responses included incorrect information in their justifications. For example, "$f$ has an inflection point at $x = 6$ because $f''(6) = 0$ and $f'$ changes from increasing to decreasing at $x = 6$."<br>• Many responses failed to use calculus to justify, stating only that $x = 2$ and $x = 6$ are inflection points "because $f$ changes concavity at these points."<br>• Some responses concluded $x = 0$ and $x = 4$ were inflection points of the graph of $f$ because $f'(0) = f'(4) = 0$. | • The graph of $f$ has inflection points at $x = 2$ and at $x = 6$ because $f'(x)$ changes from decreasing to increasing at $x = 2$ and changes from increasing to decreasing at $x = 6$.<br>• $f''(6)$ is undefined.<br>• $x = 2$ and at $x = 6$ are inflection points of $f$ because the signs of the slopes of $f'(x)$ change from negative to positive or vice versa at these points.<br>• $x = 0$ and $x = 4$ are critical points of the graph of $f$ because $f'(0) = f'(4) = 0$. |
| • In part (c) responses sometimes presented a vague or poorly-communicated argument regarding shifts of a graph to explain why $g$ was decreasing on the interval $0 \le x \le 5$. | • $g'(x) = f'(x) - 1$<br>$g'(x) \le 0$ when $f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$. The graph of $g$ is decreasing on $0 \le x \le 5$ because $g'(x) \le 0$ on this interval. |
| • In part (d) responses often presented a local argument that there was a minimum of $g$ at $x = 5$, e.g., providing only the statement that $g'(x)$ changes from negative to positive at $x = 5$, without stating that $x = 5$ is the only critical point in the interval $0 \le x \le 7$.<br>• Many responses used an incorrect value of $f(0) = 3 - 2\pi$ found in part (a) and the Candidates<br><br>Test to conclude that the absolute minimum of $g$ on $0 \le x \le 7$ was $f(0) = 3 - 2\pi$. These responses did not recognize the contradiction with their correct answer | • Because $g'(x)$ changes from negative to positive at $x = 5$, $g(5)$ is a local minimum value. Because $x = 5$ is the only value of $x$ in $0 \le x \le 7$ where $g'(x)$ changes sign, $g(5)$ is the absolute minimum value of $g(x)$ in $0 \le x \le 7$.<br>• Because $g$ is continuous and decreasing for $0 \le x \le 5$, the absolute minimum of $g(x)$ on the<br><br>interval $0 \le x \le 7$ must occur at a critical point or at the right endpoint, $x = 7$. Because $g(x)$ is increasing |

| | |
|---|---|
| in part (c) that the continuous function $g$ was decreasing on $0 \le x \le 5$. | for $5 \le x \le 7$, the absolute minimum of $g$ in the interval $0 \le x \le 7$ must occur at the critical point $x = 5$. |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Teachers could provide practice with problems involving a graphical stem, requiring students to appeal directly to the given graph using the name of the given function in each reference. Responses must not be ambiguous when it comes to which function or graph is being discussed.

- Teachers could provide specific examples of correct global arguments that can be used to justify absolute extrema and have students practice giving both these global arguments and arguments using the Candidates Test. It would be helpful to provide multiple opportunities for students to identify the differences between local and global arguments and when each is appropriate.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- In AB3/BC3, students needed to identify mathematical information in the given graph (skill 2.B). Noting that this was the graph of $f'$, the derivative of $f$, was necessary for students to identify the Fundamental Theorem of Calculus as the appropriate theorem to apply (skill 3.B). Perhaps most important in this question, students needed to be able to describe the relationships among different representations of functions and their derivatives (skill 2.E). A table of instructional strategies to help develop skills within Mathematical Practice 2, Connecting Representations, is provided on page 216 of the CED. Regular practice with the sample activity provided for skill 2.E would help students to develop the pre-reading skills needed to set up successful solutions to questions similar to AB3/BC3.

- In all parts of this question, students needed to be able to interpret the behavior of a function, $f$, based on the graph of its derivative, $f'$ (see Topic 6.5, CED and AP Classroom).

    - Video 1 in Topic 6.5 helps students to recognize various presentations of the same mathematical task: analysis of a function based on information about its derivative function. This video considers three related examples of multiple-choice questions: In the first, we are given the graph of $f$ and an accumulation function, $g$, whose derivative is $f$; in the second, we are given the graph of $f$, which is identified as the accumulation function itself; and in the third, an initial value question is framed within the context of particle motion, given an expression for velocity. If the velocity expression in the third example is understood to be analogous to the graph of $f'$ in AB3/BC3, the approach in the video is the same as the approach to part (a).

    - Video 2 in Topic 6.5 begins by reviewing important information from Topics 5.6 and 5.9 concerning conclusions about the concavity and points of inflection of a function, $f$, that can be reasoned based on information about its derivative, $f'$. This information is needed to answer part (b) of AB3/BC3.

    - Under "More Resources" in Topic 6.5 on AP Central, you may find a Lesson (for teachers) and Handout (for students) to develop understanding and mastery of relevant topics and skills: *Justifying Behavior of* $f(x)$ *from a graph of* $f'(x)$.

## Question AB4/BC4

**Topic:** Modeling Rates with IVT and Riemann Sum
**Max Score:** 9
**Mean Score: AB4** 2.97
**Mean Score: BC4** 4.28

### *What were the responses to this question expected to demonstrate?*

In this problem the melting of an ice sculpture can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is a twice-differentiable function $r(t)$ measured in centimeters, with time $t$, $0 \le t \le 12$, in days. Selected values of $r'(t)$ are provided in a table.

In part (a) students were asked to approximate $r''(8.5)$ using the average rate of change of $r'$ over the interval $7 \le t \le 10$ and to provide correct units. A correct response should estimate the value using a difference quotient, drawing from the data in the table that most tightly bounds $t = 8.5$. The response should include units of centimeters per day per day.

In part (b) students were asked to justify whether there is a time $t$, $0 \le t \le 3$, for which the rate of change of $r$ is equal to $-6$. A correct response will use the Intermediate Value Theorem, first noting that the conditions for applying this theorem are met—specifically that $r'(t)$ is continuous because $r(t)$ is twice-differentiable and that $-6$ is bounded between the values of $r'(0)$ and $r'(3)$ given in the table. Therefore, by the Intermediate Value Theorem, there is a time $t$ such that $0 < t < 3$, with $r'(t) = -6$.

In part (c) students were asked to use a right Riemann sum and the subintervals indicated by the table to approximate the value of $\int_0^{12} r'(t)\,dt$. A correct response should present the sum of the four products $\Delta t_i \cdot r'(t_i)$ drawn from the table and obtain an approximation value of $-51$.

In part (d) students were told that the height of the cone decreases at a rate of $2$ centimeters per day and that at time $t = 3$ the radius of the cone is $100$ cm and the height is $50$ cm. They are asked to find the rate of change of the volume of the cone with respect to time at time $t = 3$ days. A correct response will use the product and chain rules to differentiate the given function for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, and then evaluate the resulting derivative using values $r = 100$, $h = 50$, $\left.\frac{dh}{dt}\right|_{t=3} = -2$, and $\left.\frac{dr}{dt}\right|_{t=3} = -5$ (from the table) to obtain a rate of $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses successfully approximated $r''(8.5)$ using the average rate of change of $r'$ over the given interval, although many did not report any units or reported incorrect units. Most responses simplified their numerical answers, and this sometimes resulted in incorrect final values.

Responses had difficulty with part (b), often failing to bound $-6$ between $r'(0)$ and $r'(3)$ and/or failing to justify the continuity of $r'$. However, most responses did seem to realize that they needed to use the Intermediate Value Theorem to answer this prompt.

In part (c) most responses show a strong understanding of how to compute a right Riemann sum, although some struggled to show the required setup, and there were many errors in attempts to simplify the numerical sum.

In part (d) some responses did recognize the need for the product rule and did correctly apply both the product and chain rules to find the correct derivative of the volume with respect to time. However, many responses failed to recognize that the volume of the cone was a function of two independent variables requiring the product rule in order to find $\frac{dV}{dt}$. Instead, some made an incorrect assumption that $r = 2h$ and so reduced the function for the volume of the cone to a function of only one variable, seeming to eliminate the need to use the product rule. Other responses assumed either $r$ or $h$ was a constant, and some merely wrote expressions derived from no rules, e.g., $\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} \cdot \frac{dh}{dt}$.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) the most frequent gap in knowledge was in finding the correct units. Many responses reported units of $\text{cm}^2/\text{day}$ or $\text{cm}/\text{day}$ or failed to include any units.<br><br>• Several responses apparently did not understand the question; they found a correct approximation for $r''(8.5)$ but then used this value as the slope of a linear equation. | • $r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 \text{ cm/day} - (-4.4) \text{ cm/day}}{3 \text{ days}}$ $= 0.2$ centimeter per day per day |
| • In part (b) many responses failed to explain why the function $r'$ was continuous or discussed the continuity of $r$, not $r'$.<br><br>• Many responses failed to clearly bound the value $-6$ between $r'(0)$ and $r'(3)$.<br><br>• Most responses had difficulty providing clear communication, instead using the vague terms "it" and "the function." | • Because $r$ is twice-differentiable, $r'$ is differentiable and, therefore, continuous.<br><br>• $r'(0) = -6.1 < -6 < -5.0 = r'(3)$<br><br>• By the Intermediate Value Theorem, because is $r'$ continuous on $[0, 3]$ and $r'(0) < -6 < r'(3)$, there must exist a value $t$, $0 \le t \le 3$, such that $r'(t) = -6$. |
| • In part (c) some responses assumed the width of each rectangle was $\frac{12}{4} = 3$.<br><br>• Many responses presented errors in attempting to simplify the Riemann sum.<br><br>• Some responses concluded the value of the Riemann sum must be positive and reported a final answer of $51$, in spite of correctly calculating $-51$. | • $\int_0^{12} r'(t)\, dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$ $= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$ |
| • In part (d) many responses did not use the product rule. | • $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$ |

| | |
|---|---|
| ○ Some assumed $r = 2h$, obtaining $\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$.<br>○ Some assumed either $r$ or $h$ was a constant.<br>• Some responses failed to realize that the height decreasing at a rate of $2$ cm per day meant $\frac{dh}{dt} = -2$ and instead used the value $\frac{dh}{dt} = 2$. | • $\left.\frac{dV}{dt}\right\|_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Whenever appropriate, teachers should include units, providing students experiences in dimensional analysis.
- Teachers could provide students with a variety of examples and experiences with related rates problems—some of which require differentiating functions of one variable, others needing to differentiate functions of two independent variables. Students need practice recognizing when no functional relationship exists between variables (requiring the use of the product or quotient rule to differentiate) and when there is such a functional relationship that can be used to simplify an expression before differentiating.
- Writing succinct, clear mathematical justifications is a skill that students need to practice. Teachers should provide opportunities for students to practice writing such justifications clearly and correctly, showing students why certain parts of a justification are mathematically necessary.
- Although teachers are wise to require simplification of algebraic and numerical expressions in their own classrooms, they should remind students this is not necessary on the free-response section of the AP Exam.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- Skills associated with Mathematical Practice 4, Communication and Notation, were especially important in AB4/BC4. Suggested instructional strategies to develop these skills begin on page 219 of the CED. The strategy “Error Analysis,” which is described on page 206 of the CED, would be helpful in developing skill 4.B (Use appropriate units) and skill 4.A (Use precise mathematical language). Increased proficiency in skill 4.B would have helped students who missed the units in part (a). In part (b) vague references to “it” or “the function” are examples of imprecise mathematical language that impaired performance. Error analysis would also be useful in developing mastery of skill 3.C (Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied), which would have improved performance on part (b).
- Part (d) assesses understanding of related rates problems (see Topics 4.4 and 4.5 in the CED and on AP Classroom) and application of differentiation rules (see Units 2 and 3 in the CED and on AP Classroom).
  - Although the question gives the appropriate formula for the volume of a cone in terms of its radius and height, some students made an incorrect attempt to simplify the expression by assuming that the relationship between $r$ and $h$ when $r = 100$ cm and $h = 50$ cm was generally true ($r = 2h$). This error might represent a case of mistakenly thinking that this problem is analogous to one involving a draining conical tank, whose radius and height are confined to a given ratio by the tank, a contextual circumstance that is not true in the case of the melting ice sculpture. In Topic 4.5 on AP Classroom (and in the CED), a Lesson (for teachers) and Handout (for students) entitled *Analyzing Problems Involving Related Rates* is provided. This resource might help to develop the understanding and skills needed to solve related rates problems.
  - Video 2 in Topic 4.5 on AP Classroom provides examples of related rates problems involving volume. The first example in this video is of a volume whose shape is confined by a cubical tank and the second example is of a

conical pile of sand, whose shape is not confined by a tank, but for which it is given that $r = h$. Note that nothing similar is given in AB4/BC4. In this case, pairing this video with a review of AB4/BC4 part (d) offers an opportunity to explore with students the myriad variations of conditions that might apply to a given related rates problem.

## Question AB5

**Topic:** Differential Equation with Slope Field
**Max Score:** 9
**Mean Score:** 3.11

### *What were the responses to this question expected to demonstrate?*

In this problem students were given a differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ and told that $y = f(x)$ is the particular solution to the equation with initial condition $f(1) = 2$. They are also told that $f$ is defined for all real numbers.

In part (a) a portion of the slope field for this differential equation is shown, and students were asked to sketch the solution curve through the point $(1, 2)$. A correct response will draw a curve that follows the indicated slope segments in the first and second quadrants, through the point $(1, 2)$, with minimum and maximum points occurring at horizontal line segments on the slope field.

In part (b) students were asked to write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$ and to use that equation to approximate $f(0.8)$. A correct response would use the given differential equation to find the slope of the tangent line, $\left.\frac{dy}{dx}\right|_{(x,\,y)=(1,\,2)} = \frac{3}{2}$, then use this slope and the given point to find a tangent line equation of $y = 2 + \frac{3}{2}(x-1)$. Additionally, the response should substitute $x = 0.8$ in the tangent line equation to obtain an approximation of $f(0.8) \approx 1.7$.

In part (c) students were told that $f''(x) > 0$ for $-1 \le x \le 1$ and asked to reason whether the approximation found in part (b) is an over- or underestimate for $f(0.8)$. A correct response will reason that $f''(x) > 0$ on $-1 \le x \le 1$ means $f$ is concave up on $-1 \le x \le 1$; therefore, the tangent line lies below the graph of $y = f(x)$, and the approximation is an underestimate of $f(0.8)$.

In part (d) students were asked to use separation of variables to find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 2$. A correct response should separate the variables, integrate, use the initial condition $f(1) = 2$ to determine the value of the constant of integration, and arrive at the solution of $y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) a large majority of the students demonstrated a correct understanding of the relationship between a slope field and a solution curve for a differential equation through a particular point, although a small number of the responses failed to provide enough accuracy in the presented solution curve.

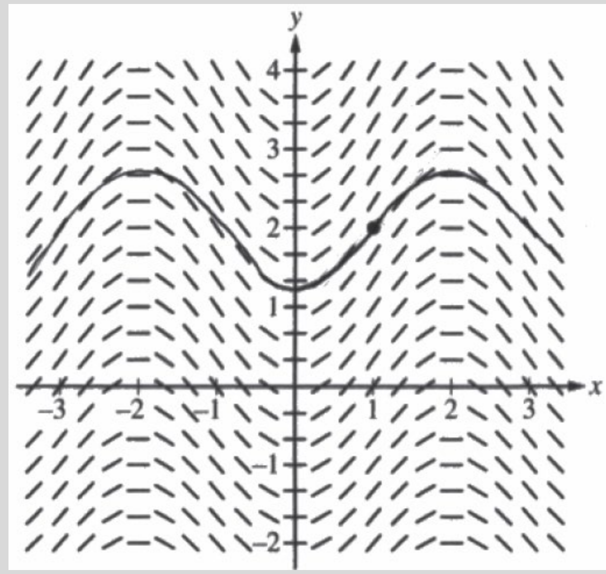
In part (b) most responses recognized the need to evaluate the given differential equation at the point $(1, 2)$ in order to find the slope of the tangent line and then wrote the tangent line equation in point-slope form. Nearly all of the responses that presented a correct tangent line equation went on to correctly use the equation to approximate $f(0.8)$. There were some errors in simplification when finding the slope or the approximation (errors that could have been avoided by not simplifying).

Part (c) was the most challenging part of this problem for the respondents. Many responses presented the wrong conclusion (overestimate) because they did not use the given information about the second derivative of $f$. Other responses that presented the correct conclusion often supported their conclusion with ambiguous, incomplete, or incorrect reasoning.

In part (d) most students attempted to separate the variables as directed. Some responses were not entirely successful in separating because of copy errors and/or mishandling the constant. Many responses had difficulty finding the antiderivative of

$\sin\left(\frac{\pi}{2}x\right)$ but did include the constant of integration appropriately and used the initial condition $(x, y) = (1, 2)$. Algebra errors were quite common in each step which meant that not very many responses presented a correct solution in the end.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) the most common misconception was that a slope field is merely a single (tangent) line through a point. A few responses drew curves that were too linear and included points that were not differentiable (for example, cusps), and a few responses drew families of solution curves, usually including the requested particular solution curve. |   |
| • In part (b) several responses felt the need to find the second derivative of $f$ in order to find the slope of the tangent line. Others thought the differential equation must be solved (via separation of variables) in order to find the tangent line slope.<br><br>• Some responses used poor communication, writing $f(0.8) = y - 2 = \frac{3}{2}(x - 1)$. | • $\left.\frac{dy}{dx}\right\|_{(x,\, y)=(1,\, 2)} = \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$<br>An equation for the tangent line is $y = 2 + \frac{3}{2}(x - 1)$.<br><br>• The tangent line approximation is $f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$. |
| • In part (c) many responses failed to grasp a connection between the second derivative (concavity) and whether a tangent line approximation is an over- or underestimate.<br><br>• Many responses used ambiguous terms, such as “it,” “the function,” “the curve,” or “the graph,” without a clear indication of to which of $f$, $f'$, $f''$, or the tangent line the term applied. | • Because $f''(x) > 0$ for $-1 \le x \le 1$, $f$ is concave up on this interval. Therefore, the tangent line at $x = 0.8$ lies below the graph of $y = f(x)$, so the tangent line approximation must be an underestimate of $f(0.8)$. |
| • In part (d) many responses were unable to successfully antidifferentiate $\sin\left(\frac{\pi}{2}x\right)$ and/or $\frac{1}{\sqrt{y+7}}$. | • $\int \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$<br>$\int \frac{1}{\sqrt{y+7}}\, dy = 2\sqrt{y+7} + C$ |

**_Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?_**

- Teachers could continue to emphasize some fine tuning on drawing solution curves—making sure the maximum and minimum values on the curve correspond to horizontal slope segments on the slope field. Teachers should also emphasize the difference between a family of solutions and a particular solution.

- Teachers can remind students of the need for precise language and provide opportunities for practicing interpretations and explanations. Vague use of terms, such as "the function," or pronouns, such as "it," must be discouraged. Students must be encouraged to provide clear, unambiguous written explanations appropriately referencing each function.

- Students need a lot of practice with solving separable differential equations, particularly those requiring substitution to find antiderivatives. Teachers should emphasize the correct time to include the constant of integration and should have students practice solving for the solution of such differential equations.

- Teachers should continue to find opportunities to reinforce and practice prerequisite skills, such as the algebra used to solve for a particular solution to a separable differential equation.

**_What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?_**

- Slope fields are considered in Topic 7.3 in the CED and on AP Classroom. AP Daily Video 2 for Topic 7.3 on AP Classroom carefully develops understanding and skills associated with this topic. The presenter gives helpful tips about using appropriate graphing techniques (skill 4.D), such as making sure that the sketch passes through channels on the slope field, where possible, and that the sketch goes from edge to edge.

- Past exam questions are excellent resources for teaching and learning (see "Model Questions" on page 208 of the CED). In part (b) of 2022 AB4/BC4, some responses presented mistakes that were also common in part (b) of 2017 AB4/BC4 ("the potato problem"). Both questions feature the direct use of the given differential equation to find the slope of the tangent line, which students sometimes complicated in ways that uncovered gaps in understanding.

- Similar to AB3/BC3, vague references to "it" or "the function" are examples of imprecise mathematical language that impaired performance in AB5 part (c). Error analysis (page 206 of the CED) and other strategies involving focused student feedback on one another's writing can help to develop communication skills.

- Some responses to part (d) would have benefited from additional practice with application of differentiation rules. Topic 2.5 on AP Classroom includes a Lesson (for teachers) and Handout (for students) entitled *Categorizing Functions for Derivative Rules.*

# Question AB6

**Topic:** Particle Motion—Velocity-Acceleration-Speed-Position
**Max Score:** 9
**Mean Score:** 3.04

### *What were the responses to this question expected to demonstrate?*

In this problem, for time $t > 0$, particle $P$ is moving along the $x$-axis with position $x_P(t) = 6 - 4e^{-t}$. A second particle, $Q$, is moving along the $y$-axis with velocity $v_Q(t) = \frac{1}{t^2}$ and position $y_Q(1) = 2$ at time $t = 1$.

In part (a) students were asked to find the velocity of particle $P$ at time $t$. A correct response would find the derivative of the given position function, $v_P(t) = 4e^{-t}$.

In part (b) students were asked to find the acceleration of particle $Q$ at time $t$ and then to find all times ($t > 0$) when the speed of particle $Q$ is decreasing. A correct response should recognize that the acceleration of the particle is the derivative of the velocity, $a_Q(t) = v_Q{}'(t) = \frac{-2}{t^3}$, then observe that for all times $t > 0$ this acceleration is negative and the given velocity $\frac{1}{t^2}$ is positive. Therefore, the acceleration and velocity of particle $Q$ have opposite signs and thus the speed of this particle is decreasing for all $t > 0$.

In part (c) students were asked to find the position of particle $Q$ at time $t$. A correct response should integrate the given velocity function, $\int_1^t v_Q(s)\,ds = \int_1^t \frac{1}{s^2}\,ds$, and add the given initial position, $y_Q(1) = 2$, to obtain a position function of $y_Q(t) = 3 - \frac{1}{t}$.

Lastly, in part (d) students were asked to reason which particle would eventually be farther from the origin as the time $t$ approaches infinity. A correct response should evalute the limits, as $t \to \infty$, of the position functions of both particles, $\lim_{t\to\infty} x_P(t) = 6$, and $\lim_{t\to\infty} y_Q(t) = 3$. Because $6 > 3$, particle $P$ would eventually be farther from the origin than would be particle $Q$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses recognized the need to take the derivative of the position function in order to find the velocity of particle $P$. A majority of the responses were also able to take the derivative correctly.

In part (b) most responses successfully found the acceleration of particle $Q$ by correctly differentiating its velocity function. A majority of the responses realized that whether the speed of this particle at any time was increasing or decreasing depended on the sign of the particle's acceleration at that time, but many failed to also consider the sign of the particle's velocity at that time.

Performance in part (c) was quite good. Most responses recognized the need to integrate the given velocity function of particle $Q$ in order to find its position function, and most also integrated correctly. Most responses that found an antiderivative went on to include the initial condition, $y_Q(1) = 2$, and, therefore, obtained the correct position function.

In part (d) many responses recognized the need to identify the end behavior of the position functions for both particles, although several had difficulties correctly communicating their analysis. A large number of responses tried to answer this question based on reasoning related to either the velocity or acceleration of the particles rather than considering the position functions.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) there were few misconceptions observed. Some responses attempted to use the power rule to differentiate the function $x_P(t) = 6 - 4e^{-t}$ and obtained $v_P(t) = te^{-t-1}$.<br><br>• Some responses contained communication errors, such as $v_P(t) = x_P(t) = 4e^{-t}$ or $v_P(1) = 4e^{-t}$. | • $v_P(t) = x_P{}'(t) = 4e^{-t}$ |
| • In part (b) the most common misconception was that sign of the acceleration alone determines whether a particle's speed is increasing or decreasing. | • The speed of particle $Q$ is decreasing for all $t > 0$ because $a_Q(t)$ and $v_Q(t)$ have opposite signs for all $t > 0$. |
| • In part (c) a very common error was presenting $\int_1^t \frac{1}{t^2}\,dt$ with no indication of how to antidifferentiate and evaluate this intergral. | • $\int_1^t \frac{1}{s^2}\,ds = -\frac{1}{s}\Big\vert_1^t = -\frac{1}{t} + 1$ |
| • In part (d) many responses treated infinity as a numerical value, writing $6 - 4e^{-\infty}$ and/or $3 - \frac{1}{\infty}$.<br><br>• Many responses presented arguments based on "the growth," velocity, or acceleration of the particles. | • $\lim_{t\to\infty} 6 - 4e^{-t} = 6 - 4(0) = 6;\ \lim_{t\to\infty} 3 - \frac{1}{t} = 3 - 0 = 3$<br><br>• Because $\lim_{t\to\infty} x_P(t) = 6 > \lim_{t\to\infty} y_Q(t) = 3,$ particle $P$ will eventually be further from the origin. |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Students would benefit from additional review of correct notation when using variables as the limits of integration in evaluating an integral using the Fundamental Theorem of Calculus, e.g., $\int_1^t \frac{1}{s^2}\,ds$ instead of $\int_1^t \frac{1}{t^2}\,dt.$
- Teachers should be careful in their presentation of the idea of infinity and the use of the symbol $\infty$. Students need to see that it is not appropriate to perform arithmetic with infinity or to use infinity as a number; the use of correct limit notation instead should be stressed.
- Students would benefit from a review of interval notation—in particular, the differences in open, closed, and half-open/closed intervals and when each is appropriate. Also, it is never correct to present the symbol $\infty$ next to a square bracket.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- Although responses demonstrated widespread understanding of the appropriate relationships among the position, velocity, and acceleration functions in parts (a), (b), and (c), mean scores were diminished by communication errors. In part (a) some responses presented strings of equal signs, ultimately linking expressions that are not, in fact, equal. In part (c) it was possible to not earn the first point because of incorrectly using the same variable as a limit and integrand function, although recovery from this notational error was possible if the incorrect expression was followed by an attempt at integration. In part (d) substituting the infinity symbol for $t$ was a notational error that resulted in the response being ineligible to earn the last point. Instructional strategies for developing skill 4.C (Use appropriate mathematical symbols and notation) can be found on page 220 of the CED. Descriptions of "Error Analysis," "Match Mine," and "Notation Read Aloud" can be found on pages 206, 208, and 209, respectively, of the CED.

- In part (b) some responses seemed to reveal conceptual gaps regarding how to use the signs of acceleration and velocity to reason about the speed of a particle at a given time. Video 1 in Topic 4.2 on AP Classroom presents a clear introduction to the relevant concepts and how to apply them.

## Question BC2

**Topic:** Parametric Particle Motion—Speed-Position-Distance
**Max Score:** 9
**Mean Score:** 5.22

### *What were the responses to this question expected to demonstrate?*

In this problem a particle is moving along a curve in the $xy$-plane with position $(x(t), y(t))$. At time $t = 4$, the particle is at the point $(1, 5)$. The particle moves so that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$.

In part (a) students were asked to find the slope of the line tangent to the path of the particle at time $t = 4$. A correct response would provide the setup $\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)}$ and evaluate to find the slope is $\frac{\ln 18}{\sqrt{17}}$ or, using a calculator, $0.701$.

In part (b) students were asked to find the speed of the particle and the acceleration vector of the particle, both at time $t = 4$. A correct response would indicate that the speed of the particle at this time is $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.0353$ and the acceleration is $a(4) = \langle x''(4), y''(4)\rangle = \langle 0.970, 0.444\rangle$. (Either or both of these answers could be provided without use of the calculator as $\sqrt{17 + (\ln 18)^2}$ or $\left\langle \frac{4}{\sqrt{17}}, \frac{4}{9}\right\rangle$, respectively.)

In part (c) students were asked to find the $y$-coordinate of the particle's position at time $t = 6$. A correct response would integrate the rate of change of the particle's $y$-position, $\frac{dy}{dt} = \ln\left(2 + t^2\right)$, from time $t = 4$ to time $t = 6$, then add the initial condition $y(4) = 5$ to find a $y$-coordinate of the particle's position of $11.571$.

And in part (d) students were asked to find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$. A correct response would provide the calculator setup of the integral of the particle's speed over this time interval, then evaluate to find a total distance of $12.136$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses recognized that the slope of the line tangent to the particle's path can be determined by evaluating the quotient $\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)}$ and were able to set up and correctly evaluate this quotient using their calculators.

In part (b) responses generally correctly presented and evaluated the expression for the speed of the particle at time $t = 4$, $\sqrt{(x'(4))^2 + (y'(4))^2}$. Many responses were also able to provide the correct components of the acceleration vector, $a(4) = \langle x''(4), y''(4)\rangle$, and evaluate these components using their calculators. Several responses attempted to find these derivatives symbolically rather than by using their calculators. The symbolic approach tended to be less successful because it often resulted in errors using the power rule or chain rule, errors that were avoided when using the calculator. Other symbolic approach errors included missing parentheses and poor communication in which symbolic expressions were equated with numerical values (such as $y''(t) = 0.444$).

In part (c) most responses appeared to understand conceptually the need to consider $\int \ln\left(2 + t^2\right) dt$, but frequently responses were unable to move beyond this idea.

In part (d) most responses indicated a conceptual understanding that the total distance traveled by the particle over a time interval is found by integrating the particle's speed over that time interval, but frequently there was a disconnect in considering the appropriate definite integral.

### ***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) responses frequently inverted the quotient and presented slope $= \frac{x'(4)}{y'(4)}$.<br>• Many responses indicated a misunderstanding of the difference between a function and a function evaluated at a specific value, often equating the two. For example, $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\ln\left(2 + 4^2\right)}{\sqrt{1 + 4^2}}$. | • $\left.\frac{dy}{dx}\right\rvert_{t=4} = \frac{y'(4)}{x'(4)} = 0.701$<br>• $\frac{dy}{dx} = \frac{y'(t)}{x'(t)};\ \frac{y'(4)}{x'(4)} = 0.701$ |
| • In part (b) some responses presented symbolic expressions for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ but neglected to evaluate these expressions at $t = 4$. | • $a(t) = \left\langle \frac{t}{\sqrt{1+t^2}}, \frac{2t}{2+t^2} \right\rangle$;<br>$a(4) = \left\langle \frac{4}{\sqrt{1+16}}, \frac{8}{2+16} \right\rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ |
| • In part (c) responses frequently presented an indefinite integral, $\int \ln\left(2 + t^2\right)\,dt$, and also failed to add the initial position $y(4)$.<br>• Some responses failed to include the differential, e.g., $\int_4^6 \ln\left(2 + t^2\right) + y(4)$, or included the wrong differential, e.g., $\int_4^6 \ln\left(2 + t^2\right)\,dy$, resulting in either an ambiguous expression or indicating a fundamental misunderstanding of integration.<br>• Many responses attempted to evaluate $\int_4^6 \ln\left(2 + t^2\right)\,dt$ using integration by parts rather than by using a calculator. | • $y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right)\,dt$<br>• $y(6) = \int_4^6 \ln\left(2 + t^2\right)\,dt + y(4)$<br>• $\int_4^6 \ln\left(2 + t^2\right)\,dt = 6.570517$ |
| • Some responses assumed this question asked for a one-dimensional arc length and presented and evaluated the integral $\int_4^6 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dt$. Some responses presented the integral $\int_4^6 \frac{dy}{dx}\,dt$ and could go no further. | • $\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 12.136$ |

| | |
|---|---|
| • Some responses assumed the total distance traveled could be determined by calculating the distances in both the $x$- and $y$-components, then used the Pythagorean Theorem, e.g., $\sqrt{\left(\int_4^6 \frac{dx}{dt}\,dt\right)^2+\left(\int_4^6 \frac{dy}{dt}\,dt\right)^2}$. | |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Teachers could improve student performance on future exams by emphasizing the use of correct notation in all circumstances. Teachers should model the correct use of an equal sign, using the symbol only to connect equal values, never as a punctuation symbol connecting multiple lines of work. Teachers should also make sure students do not use this symbol to indicate that purely symbolic expressions are equal to numerical expressions.

- Students should be encouraged to use provided function names and notation (e.g., $\frac{dy}{dt}$ and $\frac{dx}{dt}$, or $x'$ and $y'$) rather than copying function expressions, as this avoids potential copy and missing parentheses errors.

- Teachers could provide more practice using graphing calculators and provide numerous situations in which using a calculator is absolutely necessary. Students should be encouraged to use the calculator when it is permissible.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- As in AB2, issues surrounding calculator use were problematic in some cases. In addition to the resource identified in AB2 (*AP Calculus: Use of Graphing Calculators*), regularly going over the directions provided with their AP Exam booklets gives students the opportunity to practice appropriate use of technology on every assignment and assessment well in advance of exam day. In particular, emphasize the importance of showing calculator setups before calculations are made, as well as showing all other work used in solving a problem. Excerpts from the general instructions follow:

    - "You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your question, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results."

    - "Show all of your work, even though a question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods, as well as your answers. Answers without supporting work will usually not receive credit."

    - "Your work must be expressed in standard mathematical notation rather than calculator syntax."

    - "Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If you use decimal approximations in calculations, your work will be scored on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal."

- As in AB6, notational fluency emerged as an issue in BC2. Instructional strategies for developing skill 4.C (Use appropriate mathematical symbols and notation) can be found on page 220 of the CED. Descriptions of "Error Analysis," "Match Mine," and "Notation Read Aloud" can be found on pages 206, 208, and 209, respectively, of the CED.

## Question BC5

**Topic:** Area-Volume with Integration by Parts and Improper Integral
**Max Score:** 9
**Mean Score:** 4.24

### *What were the responses to this question expected to demonstrate?*

This problem provided two figures, illustrating regions $R$ and $W$ in the first quadrant, associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively.

In part (a) students were asked to find the area of region $R$. A correct response would recognize the need to compute the definite integral of $y = \frac{1}{x}$ from $x = 1$ to $x = 5$, find an antiderivative of $\ln x$, and present an area of $\ln 5$.

In part (b) students were asked to find the volume of a solid that has region $R$ as its base and whose cross sections perpendicular to the $x$-axis are rectangles with area $xe^{x/5}$. A correct response would integrate the given area function $y = xe^{x/5}$ from $x = 1$ to $x = 5$ and then proceed to use integration by parts to evaluate the integral.

In part (c) students were asked to find the volume of the solid generated when the unbounded region $W$ is revolved about the $x$-axis. A correct response would recognize this volume as $\pi$ times the improper integral of the square of the function $y = \frac{1}{x^2}$, starting at $x = 3$. The response would use correct limit notation to rewrite the improper integral with a variable upper limit, find a correct antiderivative $\left(\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3} + C\right)$, and continue the correct limit notation to find a volume of $\frac{\pi}{81}$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses evidenced understanding of the connection between area and the definite integral. Most responses were also able to successfully determine the correct antiderivative of $\frac{1}{x}$ and evaluate $\ln x\Big|_1^5$.

In part (b) a majority of the responses showed evidence of understanding the connection between cross-sectional area and volume. Many responses not only recognized the need to integrate by parts but were also successful in using this method. Many responses were uncertain about how to correctly handle the limits of integration on a definite integral evaluated using integration by parts, but correctly antidifferentiated an indefinite integral and then evaluated at the appropriate limits.

In part (c) many responses showed evidence of understanding volume by the disc method and were mostly successful in determining a correct antiderivative of the function $\frac{1}{x^n}$, even if they did not start with the correct integrand. Most responses struggled with using correct limit notation in evaluating the improper integral. Errors included absence of limit notation, initial presentation of limit notation that was subsequently dropped, and arithmetic performed with infinity.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) some responses used incorrect integrands of either $\frac{1}{x^2}$ or $\frac{\pi}{x^2}$, which may indicate some confusion between finding area and volume. | • $A = \int_1^5 \frac{1}{x}\,dx = \ln x \Big\vert_1^5 = \ln 5$ |
| • In part (b) some responses did not use the provided cross-sectional area, instead using integrands of $\frac{1}{x}e^{x/5}$, $e^{x/5}$, or $\left(xe^{x/5}\right)^2$.<br>• Several responses struggled to correctly antidifferentiate $e^{x/5}$.<br>• Many responses did not clearly label their work, making it difficult to locate specific errors in applying integration by parts.<br>• Many responses simplified incorrectly by failing to distribute the subtraction across parentheses. | • $V = \int_1^5 xe^{x/5}\,dx$<br>• Using integration by parts,<br>$u = x \quad dv = e^{x/5}\,dx$<br>$du = dx \quad v = 5e^{x/5}$<br>$\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$<br>$= 5xe^{x/5} - 25e^{x/5} + C$<br>• $V = 5e^{x/5}(x-5)\Big\vert_1^5 = 5e^1(0) - 5e^{1/5}(-4) = 20e^{1/5}$ |
| • The most frequent error in part (c) was the failure to use clear and consistent limit notation when evaluating the improper intergral. Quite a few responses omitted limit notation entirely.<br>• Many responses presented arithmetic with $\infty$, such as $\frac{1}{\infty^3}$.<br>• Some responses worked with the incorrect integral $\pi\int_3^\infty \frac{1}{x^2}\,dx$, which may have been a misconception of how to find the volume of a solid of revolution. | • $V = \pi\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty}\int_3^b \left(\frac{1}{x^2}\right)^2 dx$<br>$= \pi \lim_{b\to\infty}\left(-\frac{1}{3x^3}\Big\vert_3^b\right) = -\frac{\pi}{3}\lim_{b\to\infty}\left[\frac{1}{b^3} - \frac{1}{3^3}\right]$<br>$= -\frac{\pi}{3}\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$ |

### ***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Teachers should emphasize the differences and similarities among integrands for areas, cross-sectional volumes, and rotational volumes. Students should be encouraged to carefully read prompts to ensure they are working with the correct integrand in each situation.

- Teachers could provide significant practice integrating by parts, expecting a clear delineation of both $u$ and $dv$. Students should be required to provide unambiguous communication, including appropriate labels on intermediate steps, particularly if they are using a table. Additionally, teachers should look for ways to help students recognize situations when a particular technique of integration (substitution, by parts, etc.) is appropriate.

- Teachers could provide more demonstrations of and practice with using correct limit notation in evaluating improper integrals, making sure to carry the correct notation throughout the problem. Teachers should stress writing clear and logical mathematical work, paying particular attention to expressions involving limits and the proper placement of bounds of integration.

- Teachers should repeatedly provide opportunities for students to practice working with fractional exponents and simplifying nested algebraic expressions.

- Teachers should avoid using arithmetic with $\infty$ and not allow this on the part of their students.

- Although teachers may require simplification of algebraic and numerical expressions in their classrooms, they should remind students this is not necessary on the free-response section of the AP Exam and is perhaps costly in terms of time and errors. Teachers should provide practice AP Exams where students are encouraged not to simplify their numerical answers.

### ***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- In part (a) some responses offered evidence of confusion regarding area and volume formulas. While memorizing formulas is a time-honored practice, understanding where they come from and what they mean represents deeper and more durable learning. Videos in Topic 8.4 on AP Classroom develop the content and skills needed to answer the question in part (a). At least as important, subsequent videos developing content for later topics revisit the idea of building an integrand for an area (or a volume or a length) and integrating to determine the sum of infinitely many such components. (For example, see Video 1 from Topic 8.7.)

- On a similar note, some responses attempting to present algorithmic methods to perform integration by parts in part (b) fell short of communicating the use of the relevant integration technique. The tabular method, for example, may be used to show integration by parts, but only if the technique is made evident by clear labels. Video 1 from Topic 6.11 on AP Classroom introduces integration by parts and builds understanding by connecting the technique to the product rule for derivatives. Video 2 from Topic 6.11 offers much-needed additional practice with integration by parts. Video 3 from 6.11 illustrates how to apply integration by parts using a table of selected values for a particular function.

- In part (c) a response must write an appropriate improper integral for a solid of revolution for the unbounded region W and correctly use limits to evaluate the improper integral. Video 1 in Topic 6.13 on AP Classroom introduces improper integrals and demonstrates how to correctly use limits to evaluate a convergent improper integral. Video 2 in Topic 6.13 on AP Classroom provides additional opportunities to practice these techniques.

## Question BC6

**Topic:** Series—Interval of Convergence-Alternating Series Error Bound
**Max Score:** 9
**Mean Score:** 3.24

### *What were the responses to this question expected to demonstrate?*

In this problem a function $f$ is defined by a power series for all real numbers for which the power series converges.

In part (a) students were asked to use the ratio test to find the interval of convergence for the power series. A correct response should set up the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit of this ratio as $n \to \infty$ to find the interior of the interval of convergence. The response should then use the Alternating Series Test to determine that the series converges for both endpoints of this interval of convergence.

In part (b) students were asked to justify that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. A correct response would recognize that $\frac{1}{2}$ is the first term of the series for $f\left(\frac{1}{2}\right)$ and that the series for $f$ is alternating with terms that decrease in absolute value to $0$. Therefore, $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right|$ must be no more than the second term of the series for $f\left(\frac{1}{2}\right)$, which is $\frac{1}{24} < \frac{1}{10}$.

In part (c) students were asked to write the first four terms and the general term for an infinite series that represents $f'(x)$. A correct response would differentiate the first four given terms of the series for $f(x)$.

Finally, in part (d) students were asked to use the series from part (c) to find the value of $f'\left(\frac{1}{6}\right)$. A correct response must recognize that the series for $f'\left(\frac{1}{6}\right)$ is geometric with $a = 1$ and $r = -\frac{1}{36}$. The value of $f'\left(\frac{1}{6}\right)$ is the sum of the geometric series, $\frac{36}{37}$.

### *How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?*

In part (a) most responses were able to apply the ratio test and find the interior of the interval of convergence using proper limit notation. A majority of the responses were able to correctly determine the convergence of the series for at least one of the endpoints, and nearly all were able to compile their work into an interval of convergence.

In part (b) many responses recognized the need to find the second term of the alternating series as a bound on the error between $f\left(\frac{1}{2}\right)$ and the first term of the series, found the value of the second term, and noted this value was less than $\frac{1}{10}$. Quite a few responses also identified the series as alternating with terms that decreased to zero, a condition for applying the alternating series error bound.

In part (c) the vast majority of responses were able to find the first four terms of the series for $f'(x)$, although several responses struggled with differentiating the general term of the series for $f$ in order to find the general term for $f'$.

In part (d) some students recognized the series for $f'\left(\frac{1}{6}\right)$ as a geometric series, and those that could determine the common ratio, $r$, were successful in finding the sum of the series.

***What common student misconceptions or gaps in knowledge were seen in the responses to this question?***

| *Common Misconceptions/Knowledge Gaps* | *Responses that Demonstrate Understanding* |
|---|---|
| • In part (a) some responses had difficulty setting up the correct ratio because they could not correctly write the $(n+1)^{\text{th}}$-term of the series.<br><br>• Some responses (incorrectly) concluded that $x^2 < 1$ means $0 < x < 1$.<br><br>• Some responses did not know how to consider the endpoints of the interval of convergence, using $n = \pm 1$ rather than $x = \pm 1$.<br><br>• Some responses were unable to evaluate convergence of the series when $x = -1$ and/or when $x = 1$ because they failed to recognize these as alternating series.<br><br>• Some responses claimed the series obtained when $x = -1$ and/or when $x = 1$ were the alternating harmonic series. | • $\lim\limits_{n\to\infty}\left\lvert\dfrac{\dfrac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\dfrac{(-1)^n x^{2n+1}}{2n+1}}\right\rvert = \lim\limits_{n\to\infty}\left\lvert x^2\left(\dfrac{2n+1}{2n+3}\right)\right\rvert = \left\lvert x^2\right\rvert$<br><br>• $\left\lvert x^2\right\rvert < 1 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1$<br><br>• When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots,$ an alternating series whose terms decrease in absolute value to $0$.<br><br>• When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots,$ an alternating series whose terms decrease in absolute value to $0$.<br><br>• The interval of convergence is $-1 \le x \le 1$. |
| • Some responses did not state (or know) the conditions for applying the alternating series error bound.<br><br>• Many responses did not recognize $\frac{1}{2}$ as the first term of the series for $f\left(\frac{1}{2}\right)$ or did not know that the second term would bound the error in this case.<br><br>• Some responses attempted to reference the Lagrange error bound but could not find the bound. | • The series for $f\left(\frac{1}{2}\right)$ is alternating with terms that decrease to zero in absolute value.<br><br>• By the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most $\left\lvert\dfrac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right\rvert = \dfrac{1}{24} < \dfrac{1}{10}$. |
| • In part (c) many responses had difficulty finding the general term of the series for $f'(x)$. These responses were unable to compute the derivative with respect to $x$ of the given general term, $\dfrac{(-1)^n x^{2n+1}}{2n+1}$. | • The general term for the infinite series that represents $f'(x)$ is $(-1)^n x^{2n}$. |
| • In part (d) a very frequent error was failing to recognize the geometric series and providing an approximation of $f'\left(\frac{1}{6}\right)$ as the sum of the first few terms of the series. | • $f'\left(\frac{1}{6}\right)$ is geometric with $a = 1$ and $r = -\frac{1}{36}$.<br><br>• $f'\left(\frac{1}{6}\right) = \dfrac{1}{1-\left(-\frac{1}{36}\right)} = \dfrac{1}{\frac{37}{36}} = \dfrac{36}{37}$ |

| | |
|---|---|
| • Some responses recognized that the series for $f'\left(\frac{1}{6}\right)$ was geometric but were unable to determine $r,$ the ratio of the series. | |

***Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?***

- Teachers should provide practice in setting up the ratio when applying the ratio test to determine an interval or radius of convergence. In particular, students should practice writing term $(n+1),$ given term $n.$ Students should be required to use correct limit notation throughout the process of finding an interval or radius of convergence.

- Teachers should be diligent in using absolute values whenever appropriate and should require students to use them as well, never omitting them without explanation.

- Teachers should allow adequate class time to emphasize the conditions necessary to apply various convergence tests for a series.

- It would help to provide students with multiple representations of series to enable them to recognize special series (geometric, harmonic, etc.) and to become more fluid with notation.

***What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?***

- In part (a) the question prompted respondents to use the ratio test. Although most were able to apply the ratio test to find the interior of the interval of convergence, some had difficulty with one or more components of applying the ratio test, suggesting a thinner understanding or weaker mastery of the content and skills associated with Topics 10.8 and 10.13, two of the last topics in the course. The video associated with Topic 10.8 on AP Classroom does a nice job introducing the ratio test. Videos 1 and 2 from Topic 10.13 build on understandings of the ratio test to consider the radius and interval of convergence of power series.

- As mentioned in notes for BC2, the general instructions are a useful resource to use with students all year long. Of special relevance to part (b) of BC6 was the passage relating to justification: "Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied." In part (b) of BC6, a response needed to show that the alternating series error bound formula applies by verifying that the terms of the series for $f\left(\frac{1}{2}\right)$ decrease in absolute value to $0.$

- Understandably, in part (c) some students had difficulty deriving the general term of the series for $f'(x).$ Video 1 in Topic 10.15 on AP Classroom does a very nice job of taking students through the steps they will need to take to differentiate the general term of a power series.

- Video 1 from Topic 10.2 on AP Classroom is a helpful resource to develop the understanding of geometric series and practice the skills needed to evaluate such series. One point that is made in the video is that methods for evaluating a convergent series in AP Calculus BC are rare. In fact, if asked to find the value of a series at a point in AP Calculus BC, the correct response will involve using content and skills about geometric series.

  Beyond the scope of this course, there are other ways to evaluate series. Some of these are presented in the Unit 10 University Faculty Lecture, found on the Overview tab on AP Classroom.