Chief Reader Report on Student Responses:

2021 AP® Calculus AB/Calculus BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC):	1,344		
Calculus AB			
• Number of Students Scored	251,639		
• Score Distribution	Exam Score	N	%At
	5	44,265	17.6
	4	35,497	14.1
	3	48,524	19.3
	2	63,743	25.3
	1	59,610	23.7
• Global Mean	2.77		
Calculus BC			
• Number of Students Scored	124,599		
• Score Distribution	Exam Score	N	%At
	5	47,744	38.3
	4	20,587	16.5
	3	25,399	20.4
	2	22,655	18.2
	1	8,214	6.6
• Global Mean	3.62		
Calculus BC Calculus AB Subscore			
• Number of Students Scored	124,607		
• Score Distribution	Exam Score	N	%At
	5	58,241	46.7
	4	22,087	17.7
	3	19,911	16.0
	2	16,142	13.0
	1	8,226	6.6
• Global Mean	3.85		

* The number of students with Calculus AB subscores may differ slightly from the number of students who took the AP Calculus BC Exam due to exam administration incidents.

The following comments on the 2021 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Julie Clark of Hollins University. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB1/BC1 **Topic:** Modeling - Riemann Sum-Average Value

Max. Points: 9 **Mean Score:** AB1 2.03; BC1 3.12

What were the responses to this question expected to demonstrate?

The context of this problem is bacteria in a circular petri dish. The increasing, differentiable function f gives the density of the bacteria population (in milligrams per square centimeter) at a distance r centimeters from the center of the dish. Selected values of $f(r)$ are provided in a table.

In part (a) students were asked to use the table to estimate $f'(2.25)$ and interpret the meaning of this value in context, using correct units. A correct response should estimate the derivative value using a difference quotient, drawing from the data in the table that most tightly bounds $r = 2.25.$ The interpretation should explain that when $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of roughly 8 milligrams per square centimeter per centimeter.

In part (b) students were told that $2\pi\int_0^4 r\,f(r)\,dr$ gives the total mass, in milligrams, of the bacteria in the petri dish. They were asked to estimate the value of this integral using a right Riemann sum with the values given in a table. A correct response should multiply the sum of the four products $r_i \cdot f(r_i) \cdot \Delta r_i$ drawn from the table by $2\pi.$

In part (c) students were asked to explain whether the right Riemann sum approximation found in part (b) was an overestimate or an underestimate of the total mass of bacteria. A correct response should determine the derivative of $r \cdot f(r)$ using the product rule, use the given information that f is nonnegative to conclude that this derivative is positive and, therefore, that the integrand is strictly increasing on the interval $0 \le r \le 4.$ This means that the right Riemann sum approximation is an overestimate.

In part (d) another function, $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$, was introduced as a function that models the density of the bacteria in the petri dish for $1 \le r \le 4.$ Students were asked to find the value of k such that $g(k)$ is equal to the average value of $g(r)$ on the interval $1 \le r \le 4.$ A correct response should set up the average value of $g(r)$ as $\frac{1}{3}\int_1^4 g(r)\,dr,$ then use a graphing calculator to solve for k when setting $g(k)$ equal to this average value.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most responses demonstrated an understanding of how to estimate $f'(2.25)$ using a difference quotient, although some did not use the correct interval in their estimate. In the interpretations, responses generally understood that this value represented a rate of change, although some exhibited confusion about whether it was the bacteria population or density of the bacteria that was changing. Reporting incorrect units such as mg/cm^2 or $\left(\text{mg/cm}^2\right)^2$ was common.

In part (b) responses demonstrated a strong understanding of how to set up a right Riemann sum, but the majority provided an approximation for the wrong integral, $2\pi\int_0^4 f(r)\,dr,$ instead of $2\pi\int_0^4 r\,f(r)\,dr.$ A few responses failed to include the 2π in their estimate.

In part (c) the responses demonstrated a strong understanding of the concept that a right Riemann sum produces an overestimate of an integral for an increasing function. However, most cited $f(r)$ as the increasing function rather than $r\,f(r).$ The few responses that did correctly claim $r\,f(r)$ was increasing failed to provide adequate justification by using the product rule to determine $\frac{d}{dr}(r\,f(r)).$

In part (d) the responses generally performed well and demonstrated a correct understanding of how to compute the average value of $g(r).$ A few responses had difficulty using their graphing calculators to solve the equation to find $k,$ but this did not seem to be a major issue.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) responses had difficulty in the interpretation differentiating between density of the bacteria and the population of bacteria and reported incorrect units.	• At a distance of $r = 2.25$ cm from the center of the petri dish, the density of the bacteria is increasing at a rate of 8 mg per square cm per cm.
• In part (b) responses tended to report a right Riemann sum for $f(r)$ rather than for the requested $r f(r)$, which suggests a misunderstanding of how Riemann sums provide integral estimates.	• $2\pi\int_0^4 r f(r)\, dr = 2\pi[1 \cdot f(1) \cdot (1-0) + 2 \cdot f(2) \cdot (2-1) + 2.5 \cdot f(2.5) \cdot (2.5-2) + 4 \cdot f(4) \cdot (4-2.5)]$ $= 2\pi[1 \cdot 2 \cdot 1 + 2 \cdot 6 \cdot 1 + 2.5 \cdot 10 \cdot 0.5 + 4 \cdot 18 \cdot 1.5]$
• In part (c) responses failed to support claims that $r f(r)$ is an increasing function.	• Because f is given to be nonnegative and increasing, $\frac{d}{dr}(r f(r)) = f(r) + r f'(r) > 0,$ and therefore $r f(r)$ is strictly increasing.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers can emphasize careful reading with a variety of examples involving unusual units. Careful reading can help students understand the difference in bacteria population and density of bacteria as well as the units involved in a rate of change. Careful reading and a variety of classroom examples will also help students to set up Riemann sums for composite functions, such as $r f(r)$, to recognize the need for determining whether a composite function is increasing or decreasing and for supporting claims about composite functions with calculus concepts.

Teachers can also remind students of the need for precise language and provide opportunities for practicing interpretations and explanations. Vague use of terms such as “the function” or pronouns such as “it” must be discouraged. Students must be encouraged to provide clear, unambiguous written explanations appropriately referencing each function.

Teachers should also continue to provide plenty of practice in using graphing calculators to solve equations and encourage students to store intermediate results.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students’ broader skills. Please see page 219 of the *CED* for examples of key questions and instructional strategies designed to develop skill 4.B, *Use appropriate units of measure*. A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy “Marking the Text,” for example, may help students to identify important information in the texts that will help to determine appropriate units of measure, as well as to interpret the questions being asked in all four parts.
- AP Classroom provides videos focused on the content and skills needed to answer this question.

- The videos for topic 2.1 develop a conceptual underpinning for average rate of change over an interval and instantaneous rate of change at a point, and then to apply that understanding, including consideration of units.
- The videos for topic 6.2 demonstrate how to approximate areas of a region bound by the graph of $y = f(x)$, the x-axis, and vertical lines $x = a$ and $x = b$ using geometric and numerical methods with subintervals of equal or unequal widths, including how to apply left-, midpoint-, and right-Riemann sum and trapezoidal sum approximations. The third of these videos also develops understanding of when a Riemann sum approximation might be an underestimate or overestimate for the true area of the region.
- The videos for topic 8.1 develop the meaning of the average value of a function, how to find average value, and offer opportunities for practice, both with and without the use of a calculator for integration.

- AP Classroom also provides numerous lessons and handouts for topics 6.2 (Approximating Areas with Riemann Sums) and 8.1 (Finding the Average Value of a Function on an Interval). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice using the product rule, try entering the keywords “product rule” in the search bar, then selecting the drop-down menu for “Resource Library.” When you filter for “Classroom-Ready Materials,” you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question AB2 **Topic:** Particle Motion - Position-Acceleration-Distance

Max. Points: 9 **Mean Score:** 2.72

What were the responses to this question expected to demonstrate?

In this problem particles P and Q move along the x-axis with velocities $v_P(t) = \sin\left(t^{1.5}\right)$ and $v_Q(t) = (t - 1.8) \cdot 1.25^t$, respectively. The velocity of both particles applies for $0 \le t \le \pi$, and at time $t = 0$, particle P is at position $x = 5$, while particle Q is at position $x = 10$.

In part (a) students were asked to find the positions of both particles at time $t = 1$. A correct response should find the net change in each particle's position as the integral of their respective velocity across the interval $0 \le t \le 1$ and add this change to each particle's position at time $t = 1$.

In part (b) students were asked whether the particles were moving toward or away from each other at this time ($t = 1$). A correct response should evaluate the given velocity functions at $t = 1$ to determine the sign of each particle's velocity. This should lead to the conclusion that particle P is moving to the right while particle Q is moving to the left. In addition, a response should use the position functions found in part (a) to determine that at time $t = 1$ particle P is to the left of particle Q and, therefore, the particles are moving toward each other.

In part (c) students were asked to find the acceleration of particle Q at time $t = 1$ and whether the speed of particle Q was increasing or decreasing at time $t = 1$. A correct response should indicate that acceleration is the derivative of velocity and find the value of $a_Q = v_Q'$ at time $t = 1$ using a graphing calculator. The response should then indicate that the particle's speed is decreasing because the particle's acceleration and velocity (sign determined in part (b)) have opposite signs at this time.

Finally, in part (d) students were asked to find the total distance traveled by particle P over the entire time interval $0 \le t \le \pi$. A correct response would use a graphing calculator to determine the value of the definite integral of the speed, $\int_0^\pi |v_P(t)|\, dt$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) performance was quite good. Most responses recognized that the net change in position is given by the definite integral of the velocity and that adding the initial position to the definite integral would give the position of the particle at the requested time. However, correct notation was rarely used. Many responses were missing the differential in their integrands which was particularly problematic when the initial condition was placed after the integral: $\int_0^1 v_P(t) + 5$.

In part (b) responses generally indicated knowledge that velocities should be used to determine whether the particles were moving toward or away from each other, but many responses did not know to use the *signs* of the velocities. Some responses attempted to use the velocities and accelerations of the particles in order to make a conclusion, and others attempted to support their conclusions only with a chart or graph.

In part (c) most responses recognized the need to compare the signs of the acceleration and velocity of the particle at time $t = 1$. However, many responses failed to communicate that the acceleration of a particle is found by differentiating the velocity of the particle and had difficulty providing correct notation for a derivative evaluated at a particular time ($v_Q'(1)$). In addition, several responses incorrectly computed the acceleration of the particle by hand by trying to use the power rule to differentiate 1.25^t and/or failing to use the product rule to differentiate $(t - 1.8)1.25^t$.

In part (d) most responses recognized the need for a definite integral and many correctly integrated the speed of the particle over the correct time interval. Some responses opted to find the total area bounded by the velocity function without using the absolute value by splitting the integral into two pieces, appropriately handling the interval over which the velocity is negative. Presenting $\left|\int_0^\pi v_P(t)\, dt\right|$ (the absolute value of the definite integral of the velocity) was the most frequent mishandling of the absolute value.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) responses failed to use the initial condition or misused the initial condition by adding it to the integrand: $\int_0^1 v_P(t) + 5\, dt$.	• The position of particle P at time $t = 1$ is $x_P = 5 + \int_0^1 v_P(t)\, dt = 5.371$.
• In part (b) many responses attempted to draw a conclusion using only the signs of the particles' velocities without considering the relative initial positions of the particles at time $t = 1$.	• $v_P(1) = 0.841471 > 0$ and $v_Q(1) = -1 < 0$, so particle P is moving to the right and particle Q is moving to the left. At time $t = 1$, $x_P(1) < x_Q(1)$, so particle Q is to the right of particle P. Therefore, at this time the particles are moving toward each other.
• In part (c) the most common conceptual error was considering only the sign of the acceleration (not both acceleration and velocity) to determine whether the speed of particle Q was increasing or decreasing.	• $a_Q(1) = v_Q'(1) = 1.027$ $a_Q(1) > 0$ and $v_Q(1) < 0$ The speed of the particle Q is decreasing at time $t = 1$ because the acceleration and velocity have opposite signs.
• In part (d) the most common error was reporting the total distance traveled by particle P as the integral of the particle's velocity rather than the intergral of its speed.	• $\int_0^{\pi} \lvert v_P(t) \rvert\, dt = 1.93148$ The total distance traveled by particle P over the time interval $0 \le t \le \pi$ is 1.931.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should emphasize the importance of correct notation, in particular, the importance of the differential to close an integral expression. Teachers should also encourage students to follow up any claims with a well-written explanation of why the claim is true. (Students needed to indicate that particle P was moving to the right at $t = 1$ because $v_P(1) > 0$ and particle Q was moving to the left at $t = 1$ because $v_Q(1) < 0$.) Many students correctly identified the motion of each particle; however, they did not include the evidence that led them to that conclusion. In addition, explanations should never include the use of vague terms such as "it," "the function," "the slope," or "the graph." "It is moving to the left then because the slope is negative" does not explain how we know the particle is moving to the left.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. Please see page 220 of the *CED* for examples of key questions and instructional strategies designed to develop skill 4.C, *Use appropriate mathematical symbols and notation.* Some responses suggested the need to improve the appropriate use of differentials in part (a) and absolute values in part (c). A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy “Notation Read Aloud,” for example, may help students to accurately interpret symbolic representations. Developing skills 1.C, 1.D (p. 215), and 3.B (p. 217), which all pertain to identifying the appropriate strategy for a given situation, would also help with all parts of this question.

- AP Classroom provides videos focused on the content and skills needed to answer this question.
    - The videos for topic 4.2 build conceptual understanding of how differentiation relates position and velocity, and velocity and acceleration of a particle in motion, developing skill 1.D. The videos also consider how to determine direction of motion and whether the particle is speeding up or slowing down and are presented using a variety of representations.
    - The videos for topic 8.2 connect the particle motion relationships (straight line) learned in topic 4.2 with corresponding formulas using integration: find displacement, total distance traveled, and final position from velocity. These videos pay particular attention to helping students identify which strategy applies in a given situation and also offer practice, both with and without the use of a calculator. The second video also introduces how the sign of velocity at a point (or over an interval) can be used to determine direction of motion.
    - The videos for topic 5.3 may also be helpful in further developing the theoretical underpinnings for particle motion applications.
- AP Classroom also provides lessons and handouts for topics 8.2 (Connecting Position, Velocity, and Acceleration of Functions Using Integrals) and 5.3 (Determining Intervals on Which a Function is Increasing or Decreasing). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice deciding whether to use the power rule or the differentiation rule for an exponential function (1.C), try entering the keywords “derivative of exponential function” in the search bar, then selecting the drop-down menu for “Resource Library.” When you filter for “Classroom-Ready Materials,” you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question AB3/BC3 **Topic:** Area-Volume - Disc Method

Max. Points: 9 **Mean Score:** **AB3** 1.55; **BC3** 3.13

What were the responses to this question expected to demonstrate?

In this problem a company designs spinning toys using various functions of the form $y = cx\sqrt{4 - x^2}$, where c is a positive constant. A graph of the region in the first quadrant bounded by the x-axis and this function for some c is given and students were told that the spinning toys are in the shape of the solid generated when this region is revolved around the x-axis. Both x and y are measured in inches.

In part (a) students were asked to find the area of the region in the first quadrant bounded by the x-axis and the region $y = cx\sqrt{4 - x^2}$ for $c = 6.$ A correct response will set up the definite integral $\int_0^2 6x\sqrt{4 - x^2}\, dx$ and use the method of substitution to evaluate the integral to obtain an area of 16.

In part (b) students were told that for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}.$ They were also told that for a particular spinning toy the radius of the largest cross-sectional circular slice is 1.2 inches and were asked to find the value of c for this particular spinning toy. A correct response will solve $\frac{dy}{dx} = 0$ to find that the largest radius occurs when $x = \sqrt{2}.$ Then using this value of x in the equation $y = cx\sqrt{4 - x^2} = 1.2,$ the value of c is found to be $0.6.$

In part (c) students were told that for another spinning toy, the volume is 2π cubic inches. They were asked to find the value of c for this spinning toy. A correct response would set up the volume of the toy as the integral $\int_0^2 \pi\left(cx\sqrt{4 - x^2}\right)^2 dx,$ evaluate this integral, and set the value equal to $2\pi.$ Solving the resulting equation for c results in $c = \sqrt{\frac{15}{32}}.$

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) some responses included a definite integral with the correct integrand, but often these responses presented incorrect limits of integration or included the constant $\pi.$ Some responses clearly did not understand the difference between using an integral to find the area under a curve and the volume of a solid generated when a region is revolved around the x-axis because they squared the integrand in part (a) or did not square the integrand in part (c). Some wrote an integral for the volume in both parts (a) and (c). Some responses demonstrated a lack of notational fluency by mishandling the -2 obtained in using u-substitution to integrate.

In part (b) responses tended to be either entirely correct or did not successfully enter the problem. The responses that realized they needed to find the value of x where the function y has a critical point were generally able to find the value of x and then solve for $c.$ Many responses indicated they were confused by the question and appeared to assume they need to find an area or volume in this part as well as in parts (a) and (c).

In part (c) some responses indicated an understanding that the volume of the region is obtained by integrating area $= \pi R^2$ by setting up the correct definite integral. Many of these responses unfortunately were then unable to successfully evaluate the integral because they did not know which technique of integration to apply. Many tried to use a u-substitution, and others tried to use integration by parts without realizing that they would need to use it twice. Some responses wrote a definite integral with correct limits and the constant $\pi,$ but did not square the integrand, $cx\sqrt{4 - x^2}.$

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) responses had difficulty correctly using the method of u-substitution to evaluate the definite integral.	• Let $u = 4 - x^2$. $du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$ $x = 0 \Rightarrow u = 4 \qquad x = 2 \Rightarrow u = 0$ $\int_0^2 6x\sqrt{4-x^2}\,dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\,du = 3\int_0^4 \sqrt{u}\,du$ $= 2u^{3/2}\Big\vert_{u=0}^{u=4} = 2 \cdot 8 = 16$
• In part (b) responses failed to recognize the need to find the x-coordinate of a critical point for y. Some tried to integrate the given derivative, others tried to solve $1.2 = cx\sqrt{4-x^2}$ for c and use this in the equation $\frac{dy}{dx} = \frac{c\left(4-2x^2\right)}{\sqrt{4-x^2}}$.	• $\frac{dy}{dx} = \frac{c\left(4-2x^2\right)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$ $x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4-\left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$
• In part (c) many responses unsuccessfully tried to integrate using a u-substitution or integration by parts.	• $V = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2\left(4-x^2\right) dx$ $= \pi c^2 \int_0^2 \left(4x^2 - x^4\right)dx = \pi c^2\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big\vert_0^2$ $= \pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should make students aware that finding area of regions and volumes of solids requires the use of *definite* integrals. Finding the limits of integration is a crucial part of the setup for these problems and should be the first step. Teachers should also emphasize that when using u-substitution, either the limits of integration must also be changed, or the antiderivative must be returned to an expression in terms of the initial variable before evaluating. Perhaps practice with evaluating integrals both ways would help students understand the importance of changing the limits of integration in order to obtain the correct evaluation. Consistently requiring students to find the new limits of integration (in terms of u) before rewriting the integral in terms of u might also be beneficial.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. Please see page 215 of the *CED* for examples of key questions and instructional strategies designed to develop skill 1.D, *Identify an appropriate mathematical rule* *or procedure…* A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Scavenger Hunt," for example, may help students to self-correct and analyze their own work as they try to identify appropriate strategies for setting up and solving problems. Developing skills 1.C (p. 215) and 3.B (p. 217), which pertain to identifying the appropriate strategy for a given situation, would help with all parts of this question.

- AP Classroom provides videos focused on the content and skills needed to answer this question.
    - The videos for topic 8.4 discuss finding the area of a region between two curves, which can be applied to part (a) to find the area "under a curve." Although many responses demonstrated mastery of this content by writing the correct integral, many others wrote integrals that either indicated incomplete mastery of the appropriate equation or underdeveloped skill identifying the topic needed (skill 1.D).
    - The videos for topic 6.9 develop the required content and skills to perform the integration in part (a): identify the need to use the chain rule (skill 1.C) and correctly apply the chain rule (skill 1.E).
    - Responses to part (b) that identified the question as an optimization problem and identified a critical point were very likely to earn most or all of the points available on this part. Videos for topic 5.2 provide the theoretical underpinnings for optimization problems (the Extreme Value Theorem). Videos for topics 5.10 and 5.11 help students to recognize and solve optimization problems.
    - Videos for topic 8.9 guide students in setting up and solving volume problems like the one in part (c). Correctly setting up the equation and carrying out the corresponding squaring results in an integral for which students are more likely to select an appropriate integration technique (skill 1.C), as opposed to trying to integrate a radical expression using u-substitution.
- AP Classroom also provides lessons and handouts for topics 5.2 (Extreme Value Theorem, Global Versus Local Extrema, and Critical Points) and 6.9 (Integrating Using Substitution). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with u-substitution, try entering the keyword "u-substitution" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question AB4/BC4 **Topic:** Graphical Analysis – L'Hospital-MVT

Max. Points: 9 **Mean Score:** **AB4** 3.09; **BC4** 5.03

What were the responses to this question expected to demonstrate?

In this problem the graph of a piecewise linear continuous function f for $-4 \le x \le 6$ is provided. It is also given that $G(x) = \int_0^x f(t)\, dt.$

In part (a) students were asked to provide the open intervals on which the graph of G is concave up. A correct response would use the Fundamental Theorem of Calculus to note that $G' = f,$ and then report the two intervals where $G' = f$ is increasing.

In part (b) the function $P(x) = G(x) \cdot f(x)$ is defined and students were asked to find $P'(3).$ A correct response would use the product rule to find an expression for $P'(x),$ then use the graph of f to find numerical values of $f(3)$ and $f'(3),$ and use the Fundamental Theorem of Calculus to find $G(3)$ and $G'(3).$ The response would substitute these values into the expression for $P'(x)$ to provide the value of $P'(3).$

In part (c) students were asked to find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}.$ A correct response would use L'Hospital's Rule to find the limit after verifying that the limits of both the numerator and denominator are zero.

In part (d) students were asked to find the average rate of change of G on the interval $[-4, 2]$ and whether the Mean Value Theorem guarantees a value $c,$ $-4 < c < 2,$ with $G'(c)$ equal to this average rate of change. A correct response would determine the average rate of change as a difference quotient, $\dfrac{G(2) - G(-4)}{2 - (-4)},$ with values $G(2) = 0$ and $G(-4) = -16$ found as areas under the graph of $f.$ The response should then conclude that the Mean Value Theorem does guarantee such a value of c because $G' = f$ is differentiable and, therefore, continuous on the given interval.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) correct responses demonstrated an understanding of the connection between a function presented as a graph and another function defined using the Fundamental Theorem of Calculus. These responses also demonstrated the ability to express how the concavity of a function is related to both functions. However, a common error was failing to expressly present the connection $G' = f.$ In many responses correct intervals were presented, but the reasoning was missing or poorly stated. Responses made liberal use of terms such as "it" or "the graphs," which made their explanations difficult or impossible to interpret.

In part (b) most responses were adept at applying the product rule to find a derivative and evaluating the derivative at a point.

In part (c) most responses were able to correctly apply L'Hospital's Rule to evaluate the given limit. However, in many cases responses were unable to communicate the need for L'Hospital's Rule or to use limit notation with the ratio of derivatives.

In part (d) most responses were able to clearly present at least an attempt to find the average rate of change of $G,$ and to correctly claim that the Mean Value Theorem applied to the function G on the given interval. In addition, many responses were able to evaluate a definite integral by correctly reversing the limits of integration, which demonstrated a graphical understanding of the definite integral. Several responses failed to recognize that the given graph of $f,$ defined on $(-4, 2),$ was a visual representation of the function G' and therefore G was both continuous on $[-4, 2]$ and differentiable on $(-4, 2).$ Several responses claimed that the Mean Value Theorem did not apply because the given graph ($G' = f$) was not differentiable.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses lacked the reasoning for the intervals presented or gave inadequate reasoning for the intervals presented.	• The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$ because $G' = f$ is increasing on these intervals.
• In part (b) the lack of a connection between G and f prevented responses from finding $G(3)$ or $G'(3)$ and then correctly evaluating $P'(x)$ at $x = 3$.	• $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ Therefore, $P'(3) = G(3)f'(3) + f(3)G'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5.$
• In part (c) many responses did not provide sufficient justification for the use of L'Hospital's Rule by indicating the presence of the $\frac{0}{0}$ indeterminate form.	• $\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$ and $\lim_{x\to 2}\left(x^2 - 2x\right) = 0$ Therefore, $\lim_{x\to 2} \frac{G(x)}{\left(x^2 - 2x\right)} = 0$ is an indeterminate form of type $\frac{0}{0}$.
• In part (d) some responses found the average rate of change of G' rather than of G, or found the average value of G. • Many responses failed to include the conditions of both continuity and differentiability in justifying the use of the Mean Value Theorem.	• Average rate of change $= \frac{G(2) - G(-4)}{2 - (-4)} = \frac{0 - (-16)}{6} = \frac{8}{3}$ • Yes, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem guarantees a value c, $-4 < c < 2$, such that $G'(c) = \frac{8}{3}$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should provide ample practice with problems requiring the use of the Mean Value Theorem, Intermediate Value Theorem, and L'Hospital's Rule, and with written justifications of the application of these theorems and rules. The ability to move beyond the process of calculation to justification is crucial for success on the AP Exam. Students should practice using limit notation correctly and consistently in appropriate situations, and they should be held to a high standard with regard to correct notation throughout their study of calculus.

In addition, teachers should provide a great deal of practice with problems that require recognizing and understanding the Fundamental Theorem of Calculus. Teachers should encourage students to begin such problems by writing an expression (such as $G' = f$) that explains how the given functions are related. Teachers should continually stress the importance of clear, unambiguous explanations, avoiding the use of words such as "it" and "the graph."

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. Please see page 216 of the *CED* for examples of key questions and instructional strategies designed to develop skill 2.E, *Describe the relationships among different representations of functions and their derivatives.* A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Sentence Starters," for example, may help students to practice communication skills needed to clearly present reasoning. Identifying and applying the Fundamental Theorem of Calculus was important in every part of this question (skills 3.B and 3.D, *Identify and apply an appropriate mathematical definition, theorem, or test,* p. 217). Mastery of skill 3.C, *Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied,* (p. 217) was important in parts (c) and (d).
- AP Classroom provides videos focused on the content and skills needed to answer this question.
  - The videos for topic 5.6 develop the content and skills necessary for successful reasoning about concavity of functions on their domains (part (a)).
  - The videos for topics 6.4 and 6.7 explore the Fundamental Theorem of Calculus (FTC) as a differentiation rule for an accumulation function and in definite integrals. A complete understanding of when and how to apply the FTC is essential to success on this question.
  - The videos for topic 5.1 present how to reason with a theorem (the Mean Value Theorem), including the importance of confirming that the hypothesis conditions have been met so that the conclusion can be drawn (skill 3.C).
- AP Classroom also provides lessons and handouts for topics 5.1 (Using the Mean Value Theorem) and 5.7 (Using the Second Derivative Test to Determine Extrema). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice working with L'Hospital's Rule, try entering the keyword "L'Hospital's Rule" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question AB5 **Topic:** Implicit Differentiation - Tangent Line

Max. Points: 9 **Mean Score:** 2.72

What were the responses to this question expected to demonstrate?

In this problem $y = f(x)$ is an implicitly defined function whose curve is given by $2y^2 - 6 = y \sin x$ for $y > 0$.
In part (a) students were asked to show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$, which can be done using implicit differentiation.
In part (b) students were asked to write an equation for the tangent line at the point $(0, \sqrt{3})$. A correct response would evaluate the derivative given in part (a) at the point $(0, \sqrt{3})$ and then write the equation of a line through the given point with slope equated to the evaluated derivative.
In part (c) students were asked to find the coordinates of the point where the line tangent to the curve is horizontal for $0 \le x \le \pi$ and $y > 0$. A correct response would set the slope of the tangent line, $\frac{dy}{dx}$, equal to zero, then determine that $y \cos x = 0$ when $x = \frac{\pi}{2}$. The response should then use the given equation $2y^2 - 6 = y \sin x$ to find $y = 2$ when $x = \frac{\pi}{2}$, which results in the point with coordinates $\left(\frac{\pi}{2}, 2\right)$.
In part (d) students were asked to determine and justify whether the function f has a relative minimum, a relative maximum, or neither at the point found in part (c): $\left(\frac{\pi}{2}, 2\right)$. A correct response would use the quotient rule to find $\frac{d^2y}{dx^2}$, determine the sign of $\frac{d^2y}{dx^2}$ at the critical point $\left(\frac{\pi}{2}, 2\right)$, and conclude that f has a relative maximum at this point.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) most responses were able to correctly differentiate the given equation implicitly. Some responses ran into trouble by starting each line of their work with $\frac{dy}{dx} =$, which led to invalid statements. Some responses attempted to use integration with the differential equation in order to obtain the given function but were unsuccessful.
In part (b) the majority of responses were able to correctly find the equation of the tangent line at $(0, \sqrt{3})$. A common error was to use the expression $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$, with no numerical evaluation as the slope of the tangent line.
In part (c) most responses recognized the need to set $\frac{dy}{dx} = 0$. Some responses were able to solve the equation for the correct value of $x = \frac{\pi}{2}$, but many responses struggled with the trigonometry required in this situation. Still other responses had difficulty with the algebra required to find the correct value of $y = 2$ given that $x = \frac{\pi}{2}$.
In part (d) responses that used the second derivative of $f(x)$ to determine that there was a relative maximum at $\left(\frac{\pi}{2}, 2\right)$ were quite successful. However, there were very few responses that recognized the need to use the second derivative. Most responses attempted to use the First Derivative Test with varying results. Most did not analyze the sign of the denominator of $f(x)$ Many responses tried to justify the maximum with incorrect statements (“the derivative is increasing”) or vague terms such as “it,” “the slope,” or “the derivative.”

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses presented incorrect statements because they began each line of their implicit differentiation with $\frac{dy}{dx} =$.	• $\frac{d}{dx}\left(2y^2 - 6\right) = \frac{d}{dx}(y \sin x) \Rightarrow 4y\frac{dy}{dx} = y \cos x + \frac{dy}{dx}\sin x$ $\Rightarrow 4y\frac{dy}{dx} - \frac{dy}{dx}\sin x = y \cos x \Rightarrow \frac{dy}{dx}(4y - \sin x) = y \cos x$ $\Rightarrow \frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$
• In part (b) several responses were unable to correctly evaluate $\frac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0}$ to find the slope of the tangent line.	• $\left.\frac{dy}{dx}\right\vert_{(x,\, y)=(0,\, \sqrt{3})} = \frac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \frac{\sqrt{3}}{4\sqrt{3}} = \frac{1}{4}$
• In part (c) responses were unable to determine when $\frac{y \cos x}{4y - \sin x} = 0$ when $y > 0$. • Many responses were unable to solve $y \sin x = 2y^2 - 6$ when $x = \frac{\pi}{2}$.	• If $y > 0$, $\frac{y \cos x}{4y - \sin x} = 0 \Rightarrow y \cos x = 0 \Rightarrow x = \frac{\pi}{2}$. • When $x = \frac{\pi}{2}$ and $y > 0$, $y \sin x = 2y^2 - 6 \Rightarrow y \sin\frac{\pi}{2} = 2y^2 - 6$ $\Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0$ $\Rightarrow (2y + 3)(y - 2) = 0 \Rightarrow y = 2.$
• In part (d) most responses that used the First Derivative Test did not consider the sign of $4y - \sin x$ and/or did not clearly state that $\frac{dy}{dx}$ changed from positive to negative values at $\left(\frac{\pi}{2}, 2\right)$.	• $\frac{d^2y}{dx^2} = \frac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y \sin x\right) - (y \cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$ When $x = \frac{\pi}{2}$ and $y = 2$, $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x} = 0$ and $\frac{d^2y}{dx^2} = \frac{(7)(-2) - (0)(0)}{(7)^2} = \frac{-2}{7} < 0.$ • $y > 0$ is given. Near $\left(\frac{\pi}{2}, 2\right)$, $4y - \sin x > 0$. Thus, $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$ changes from positive to negative at $\left(\frac{\pi}{2}, 2\right)$. Therefore, f has relative maximum at this point.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should provide practice AP Exams where students are encouraged not to simplify their numerical answers because most teachers will want numeric answers simplified on a daily basis. Teachers should also remind students that tangent line equations may be written in any form—it is not necessary to convert to slope-intercept or any other form.

Teachers should require students to show all of their work, including the presentation of all equations they solve in order to find particular values. Teachers should provide many opportunities to practice verifying a statement and should make sure students understand that an equals sign is not a punctuation symbol that can be used to connect a string of unequal values. Teachers cannot provide too much practice writing verbal calculus-based justifications with precise language and correct notation.

Teachers can encourage students to be aware of what is needed in order to solve a problem and to take advantage of the given information in order to find the needed information efficiently. Teachers might encourage students to use the Second Derivative Test to determine the location of a relative maximum or minimum, particularly when a function is defined implicitly. In addition, frequent reviews of unit circle trigonometric values would be very helpful for students.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. Mathematical Practice 4 was important to successfully responding to this question. Please see pages 219 and 220 of the *CED* for examples of key questions and instructional strategies designed to develop skill 4.A, *Use precise mathematical language,* and skill 4.C, *Use appropriate mathematical symbols and notation.* A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Round Table," for example, allows students to analyze each other's work and coach their peers to avoid vague language that was problematic in some responses to this question.
- AP Classroom provides videos focused on the content and skills needed to answer this question.
  - The videos for topic 3.2 present several examples of implicit differentiation with carefully explained algebra and trigonometry, as needed in part (a). The second of these videos discusses how to find horizontal tangent lines, as needed in part (c).
  - The third video for topic 2.2 discusses how to write the equation of a tangent line (part (b)).
  - Very few students attempted the Second Derivative Test, which was the easier way to work part (d). To identify a particular strategy, such as the Second Derivative Test, students need to be comfortable with a variety of options. The video for topic 5.7 is a good way to familiarize students with the Second Derivative Test.
- AP Classroom also provides lessons and handouts for topics 3.2 (Implicit Differentiation) and 2.9 (The Quotient Rule). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with the Second Derivative Test, try entering the keywords "Second Derivative Test" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question AB6 **Topic:** Modeling with Diff Eq - Separation of Variables

Max. Points: 9 **Mean Score:** 1.75

What were the responses to this question expected to demonstrate?

In this problem a function $y = A(t)$ models the amount of medication, in milligrams, in a patient at time t hours. This function satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$, and at time $t = 0$ hours, there are 0 milligrams of medication in the patient.

In part (a) students were shown a portion of the slope field for the given differential equation and asked to sketch the solution curve through the point $(0, 0)$. A correct response would draw a single increasing, concave down curve starting at $(0, 0)$, approaching the horizontal asymptote with slopes equal to zero from below.

In part (b) students were asked to interpret the statement $\lim_{x \to \infty} A(t) = 12$ using correct units in this context. A correct response would indicate this statement means that over time the amount of medication in the patient approaches 12 milligrams.

In part (c) students were asked to use separation of variables to find the particular solution $y = A(t)$ with $A(0) = 0$. A correct response should separate the variables, integrate, and use the initial condition $A(0) = 0$ to resolve the constant of integration and arrive at the solution $A(t) = 12 - 12e^{-t/3}$.

In part (d) a second function $y = B(t)$, which satisfies $\frac{dy}{dt} = 3 - \frac{y}{t + 2}$, is introduced as a model for the amount of medication in a second patient at time t hours. At time $t = 1$ hour, there are 2.5 milligrams of medication in the second patient. Students were asked whether the amount of medication in the patient is increasing or decreasing at time $t = 1$. A correct response would use the quotient rule to compute $B''(t) = \frac{d^2y}{dt^2}$, determine that $B'(1) \neq 0$ and $B''(1) < 0$, and then would conclude the amount of medication is decreasing.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) responses were good, with the vast majority sketching a curve that passed through $(0, 0)$, was generally increasing and concave down, and approached the horizontal asymptote.

In part (b) most responses reported the correct units (milligrams), and many seemed to make the connection between limits at infinity and horizontal asymptotes. Some responses had difficulty distinguishing between the amount of medication "in" the patient and the amount of medication "administered" to the patient. Some responses had difficulty correctly phrasing the interpretation of limit, choosing to use phrases such as "maximum of 12 " or "is 12. "

In part (c) very few responses were able to present a correct solution of $y = 12 - 12e^{-t/3}$. Too many responses failed to correctly separate the variables, and many responses that did correctly separate then incorrectly reported the antiderivative of $\frac{1}{12 - y}$ as $\ln|12 - y|$. Most responses that did correctly separate the variables, antidifferentiate, and use the initial condition were unable to solve their expression for $y = A(t)$. Many were unable to work with exponential expressions correctly; common errors involved simplifying $e^{-\ln|12-y|}$ and $e^{\frac{t}{3}+C}$.

In part (d) most responses did not recognize the need to use the second derivative to determine whether the rate of change was increasing or decreasing at $t = 1$. The few responses that did attempt to find the second derivative demonstrated correct use of the quotient rule, but had difficulty correctly handling the negative sign between the terms 3 and $\frac{y}{t + 2}$. A great many responses tried to work only with the given derivative $\frac{dy}{dt} = 3 - \frac{y}{t+2}$, without realizing they did not have enough information to evaluate this function at any time other than $t = 1$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) a few responses provided two curves, one above the horizontal axis and one below.	• There should be a single slope curve through $(0, 0)$ that is increasing, concave down, and approaches the horizontal asymptote from below.
• In part (b) the most common error was interpreting $\lim_{x\to\infty} A(t) = 12$ to mean that over time the amount of medication in the patient's body *is* 12 mg.	• Over time the amount of medication in the patient's body approaches 12 mg.
• In part (c) the most common error was $\int \frac{1}{12-y}\,dy = \ln\lvert 12-y\rvert$.	• $\int \frac{1}{12-y}\,dy = -\ln\lvert 12-y\rvert$
• In part (d) most responses failed to recognize that the sign of $B''(1)$ was needed in order to determine whether $B'(1)$ was increasing or decreasing.	• $B''(t) = (-1)\frac{B'(t)\cdot(t+2) - y}{(t+2)^2}$ $B'(1) = 3 - \frac{B(1)}{3} = 3 - \frac{2.5}{3} \neq 0$ $B''(1) = -\frac{B'(1)\cdot 3 - B(1)}{3^2} = -\frac{6.5-2.5}{9} = -\frac{4}{9} < 0$ The rate of change of the amount of medication is decreasing at time $t = 1$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should expose students to conceptual examples of limit, particularly limits at infinity. Students need lots of practice with solving separable differential equations, particularly those involving logarithmic antiderivatives. Teachers should emphasize the correct time to include the constant of integration and should have students practice solving for the solution of such differential equations.

Teachers might want to provide several mixed examples asking when functions and when rates of change are increasing or decreasing so that students learn to carefully read the problem and distinguish between these two situations. Teachers can continue to nurture and reinforce prerequisite skills, such as simplifying exponential expressions and manipulating equations involving fractions.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. In this question, skill 1.B, *Identify key and relevant information to answer a question or solve a problem*, undergirds several assessed skills and content: in part (a) skill 4.D, *Use appropriate graphing techniques*; in part (b) interpreting limits at infinity; and in part (d) recognizing that the sign of $B''(1)$ was needed. Please see page 214 of the *CED* for examples of key questions and instructional strategies designed to develop skill 1.B. A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Think-Pair-Share," for example, enables the development of initial ideas that are then tested with a partner, revised, and shared with the larger group.
- AP Classroom provides videos focused on the content and skills needed to answer this question.
  - The videos for topic 7.3 provide several examples about constructing and interpreting slope fields.
  - The videos for topic 1.15 provide algebraic treatments of limits at infinity but also connect results to horizontal asymptotes in graphical representations.
  - The videos for topics 7.6 and 7.7 provide examples of solving general and particular solutions to separable differential equations and opportunities for student practice.
  - The videos for topic 5.3 present content and skills needed to determine intervals on which a function is increasing or decreasing. Note that in part (d) the lessons of topic 5.3 must be applied to B'.
- AP Classroom also provides lessons and handouts for topics 1.15 (Connecting Limits at Infinity and Horizontal Asymptotes), 7.6 (Finding General Solutions Using Separation of Variables), and 5.3 (Determining Intervals on Which a Function is Increasing or Decreasing). There are topic questions for formative assessment of each topic, as well as access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with solving separable differential equations, try entering the keyword "separable differential equations" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question BC2 **Topic:** Parametric Particle Motion - Acceleration-Distance

Max. Points: 9 **Mean Score:** 2.19

What were the responses to this question expected to demonstrate?

In this problem a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$; the position is $(-2, 5)$ at time $t = 0$.

In part (a) students were asked to find the speed and the acceleration vector of the particle at time $t = 1.2$. A correct response would show the speed setup, $\sqrt{(x'(1.2))^2 + (y'(1.2))^2}$, and the acceleration setup, $\langle x''(1.2), y''(1.2) \rangle$, and then use a graphing calculator to find both values.

In part (b) students were asked to determine the total distance traveled by the particle over the time interval $0 \le t \le 1.2$. A correct response would present the integral $\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ and determine the numerical value using a graphing calculator.

In part (c) students were asked to find the coordinates of the point (for $t \ge 0$) when the particle is farthest to the left. They were also asked to explain why there is no point at which the particle is farthest to the right. A correct response would determine that the particle changes direction when $x'(t)$ changes sign at time $t = 1$, by setting $x'(t) = 0$. Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the left-most position of the particle would be computed by adding the initial position to the net change, found by integrating the velocity function from $t = 0$ to $t = 1$, for each coordinate position of the particle. Finally, the response should argue that the particle's initial x-coordinate at time $t = 0$ is to the right of the particle's position at time $t = 1$, and from this time on, the particle is moving to the right. Therefore, there is no point at which the particle is farthest to the right.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) the majority of the responses correctly found the speed and acceleration of the particle from the given velocity vector. Although many responses presented a correct numerical answer, a common mistake was equating the numerical value to a symbolic expression. Some responses failed to provide the setup for the presented numerical answer, and a few responses confused the velocity vector with a position vector. Some responses miscopied the actual derivatives from the stem instead of using the notation $x''(1.2)$ and $y''(1.2)$.

In part (b) most responses correctly provided the definite integral of the speed, although some responses failed to obtain the correct numerical value from their graphing calculator.

In part (c) most responses realized the need to determine the location of critical points of the x-coordinate velocity function, so they set $x'(t) = 0$, and found $t = 1$. Many responses were also able to use the Fundamental Theorem of Calculus to find the position of the particle when $t = 1$. However, some responses only justified a local minimum at $t = 1$, failing to either note that this was the only critical point, or that $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for all $t > 1$. In addition, very few responses were able to explain why there was no point at which the particle was farthest to the right.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses did not present enough symbolic information to justify the work done using a graphing calculator or linked symbolic and numerical expressions such as $\sqrt{(x'(t))^2 + (y'(t))^2} = 1.271.$	• At time $t = 1.2,$ the speed of the particle is $\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271.$ • At time $t = 1.2,$ the acceleration of the particle is $\langle x''(1.2),\ y''(1.2)\rangle = \langle 6.247,\ 0.405\rangle.$
• In part (b) some responses provided general formulas such as $\int_0^{1.2} \lvert v\rvert\, dt$ rather than expressions in context and/or used the speed at $t = 1.2$ as the integrand: $\int_0^{1.2} \lvert 1.271\rvert\, dt$ or $\int_0^{1.2} \left\lvert \sqrt{(x'(1.2))^2 + (y'(1.2))^2} \right\rvert\, dt.$	• $\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009817$ The total distance traveled by the particle over the time $0 \le t \le 1.2$ is 1.010.
• In part (c) most responses did not understand the need to demonstrate that $x(t)$ increased without bound and $x'(t)$ needed to be positive (for $t > 1$) in order to show that there was no rightmost point.	• Because $x(2) = -2 + \int_0^2 x'(t)\, dt > -2 = x(0),$ the particle moves to the right of its initial position. Because $x'(t) > 0$ for $t > 1,$ the particle moves to the right for all $t > 1.$ Therefore, there is no point at which the particle is farthest to the right.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Once again, teachers could best help their students by emphasizing written explanations of their answers, in this case providing a global explanation about the sign of $x'(t)$ rather than a local argument. Symbolically, teachers should model the correct use of an equal sign, using the symbol only to connect equal values, never as a punctuation symbol connecting multiple lines of work. Teachers should also make sure students do not use this symbol to indicate purely symbolic expressions are equal to numerical expressions, e.g., $\sqrt{(x'(t))^2 + (y'(t))^2} = 1.271.$ This will require emphasizing the correct notation for evaluating an expression at a specific numerical value.

Teachers could provide more practice using graphing calculators to store functions in order to easily calculate square roots, derivatives, integrals, and sums of squares. Students should be encouraged to use given function names when presenting the expression used in the graphing calculator rather than trying to rewrite the entire function definition, as this often results in a "copy error."

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. In this question, skill 1.E, *Apply appropriate mathematical rules or procedures, with and without technology,* is important. Please see page 215 of the *CED* for examples of key questions and instructional strategies designed to develop skill 1.E. A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Collaborative Poster," for example, allows students time to analyze each other's work and explain their reasoning to others.
- AP Classroom provides videos focused on the content and skills needed to answer this question. The videos for topic 9.6 fully develop the content and skills needed to answer this question and offers opportunities for students to practice using the calculator.
- Topic 9.6 in AP Classroom also provides a link to an excellent curriculum module, *Vectors*. AP Classroom also provides topic questions for formative assessment and access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with vectors, try entering the keyword "vectors" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question BC5 **Topic:** Separable Diff Eq - Taylor Poly-Euler-Int by Parts

Max. Points: 9 **Mean Score:** 4.26

What were the responses to this question expected to demonstrate?

In this problem $y = f(x)$ is the particular solution to $\frac{dy}{dx} = y \cdot (x \ln x)$ with $f(1) = 4$ and students were told that $f''(1) = 4.$ In part (a) students were asked to write the second-degree Taylor polynomial for f about $x = 1$ and to use the polynomial to approximate $f(2).$ A correct response would determine that $f'(1) = 0$ and use this value and the given values of $f(1)$ and $f''(1)$ to write the polynomial $4 + 2(x-1)^2.$ The response would then find an approximation of $f(2) \approx 6.$

In part (b) students were asked to use Euler's method to approximate $f(2)$ using two steps of equal size starting at $x = 1.$ A correct response would use Euler's method with $\Delta x = 0.5$ to first approximate $f(1.5) \approx 4$ and then use that value with Euler's method to approximate $f(2) \approx 4 + 3\ln 1.5.$

In part (c) students were asked to find the particular solution $y = f(x)$ with initial condition $f(1) = 4.$ A correct response would use integration by parts to find $\ln|y| = \int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C,$ with $C = \ln 4 + \frac{1}{4}$ determined from the initial condition. Then solving for y results in the solution $y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}.$

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) a majority of the responses showed a familiarity with the formula for a Taylor polynomial, but some provided a generic formula without translating to the context of the given situation. Some responses centered the polynomial at $x = 0,$ and some incorrectly simplified $\ln 1$ to 1. Some responses only provided the approximation of $f(2),$ without ever writing the requested Taylor polynomial.

In part (b) most responses demonstrated some understanding of Euler's method, but many responses had difficulty communicating their use of the method correctly. This was particularly true of responses that presented the method through a table. Some responses did not use the correct value of x in their calculations – using $x = 1.5$ to calculate $f(1.5)$ in the first step and using $x = 2$ to approximate $f(2).$ Some responses displayed the step size but did not use it in their calculations, while other responses did not consider the step size in any way.

In part (c) the majority of responses demonstrated an understanding of the need to separate the variables and then to integrate both sides of the resulting equation. A majority also correctly evaluated $\int \frac{1}{y} \, dy,$ but many did not correctly evaluate $\int x \ln x \, dx.$ Some responses did not recognize the need to use integration by parts for this second integral, and others were unsuccessful in using integration by parts. Errors in attempting integration by parts included misidentifying u and $dv,$ not evaluating du and/or v correctly, and using addition rather than subtraction in the parts formula. Of those that did correctly integrate both functions, several responses did not introduce the constant of integration at the correct step or did not correctly solve the equation involving the antiderivatives for the function $y.$

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) using an incorrect center of $x = 0$: $4 + 0x + 2x^2$	• $4 + 0(x-1) + 2(x-1)^2$
• In part (b) not using the step size $\Delta x = 0.5$: $f(1.5) \approx f(1) + 4(1 \cdot \ln 1) = 4$ $f(2) \approx f(1.5) + 4(1.5 \cdot \ln 1.5) \approx 4 + 6\ln 1.5$ • Neglecting to add the result of the first step in the second step: $f(2) \approx (0.5)4(1.5 \cdot \ln 1.5) = 3\ln 1.5$	• $f(1.5) \approx f(1) + (0.5) \cdot 4 \cdot (1 \cdot \ln 1) = 4$ $f(2) \approx f(1.5) + (0.5) \cdot 4 \cdot (1.5 \cdot \ln 1.5) \approx 4 + 3\ln 1.5$ • $f(2) \approx f(1.5) + (0.5)4(1.5 \cdot \ln 1.5) \approx 4 + 3\ln 1.5$
• In part (c) many responses used the constant of integration incorrectly: $\ln y = \frac{x^2 \ln x}{2} - \frac{x^2}{4} \Rightarrow y = e^{\frac{x^2 \ln x}{2} - \frac{x^2}{4}} + C.$	• $\ln y = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$ or $y = Ce^{\frac{x^2 \ln x}{2} - \frac{x^2}{4}}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should provide numerous situations asking for Taylor polynomials, some of which do not provide an explicit formula for the function (like this one). Examples where values of some derivatives evaluated at a point are given and derivatives are expressed in notation other than f', such as $\frac{dy}{dx}$, or $\frac{d^2y}{dx^2}$ would help students gain a full understanding of how to find the coefficients for a Taylor polynomial.

Teachers may want to provide more practice applying recursive algorithms such as Euler's method. Stressing the connection between Euler's method and repeated use of tangent line approximations might help students understand the recursive equation involved. When practicing the use of Euler's method, teachers should demand clear labeling of tables and intermediate calculations and a clear indication of the final answer.

Teachers should provide plenty of practice with the technique of integration by parts and correct use of the constant of integration whenever students are asked to evaluate an indefinite integral. Teachers might consider providing more complicated examples involving separation of variables to allow students to encounter solutions involving logarithms and exponential functions in context. Teachers can continue to find opportunities to reinforce and practice prerequisite skills, such as the algebra used to solve for a particular solution to a separable differential equation.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. In this question, skill 1.E, *Apply appropriate* *mathematical rules or procedures, with and without technology* (please see *CED*, p. 215), is important. Also important on part (a) of this question are skill 2.C, *Identify a re-expression of mathematical information presented in a given representation* (please see *CED*, p. 216), and skill 3.D, *Apply an appropriate mathematical decision, theorem, or test* (please see *CED*, p. 217). A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Create Representations," for example, might be adapted to develop student understanding that increasing the degree of a Taylor polynomial approximation may result in the nth -degree polynomial approaching the graph of the original function over a given interval (LIM-8.A.2, skill 2.C).

- AP Classroom provides videos focused on the content and skills needed to answer this question.
  - The videos for topic 10.11 fully develop the content and skills needed to write the second-degree Taylor polynomial for the given function f and to use the Taylor polynomial to approximate $f(2)$.
  - The videos for topic 7.5 are careful to build the understanding of Euler's method that is more likely to result in correct work than mere memorization, which may result in errors such as those described above: not using the step size or neglecting to add the result of the first step in the second step.
  - The videos for topics 7.6 and 7.7 offer many completely explained examples and opportunities for student practice determining general and particular solutions to differential equations.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with Taylor polynomial approximations, try entering the keyword "Taylor polynomial" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.

Question BC6 **Topic:** Integral Test-Limit Comp Test-Alt Series Error Bound

Max. Points: 9 **Mean Score:** 2.25

What were the responses to this question expected to demonstrate?

In this problem the function g has derivatives of all orders for all real numbers and students were given the Maclaurin series for g.

In part (a) students were asked to state the conditions necessary to use the integral test to determine whether the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, and then to use the integral test to show that the series converges. A correct response should state that the integral test requires the function $\frac{1}{e^x}$ to be positive, decreasing, and continuous on the interval $[0, \infty)$. The response should continue by demonstrating that the improper integral $\int_0^{\infty} e^{-x}\, dx$ is finite and therefore converges, so $\sum_{n=0}^{\infty} e^{-n}$ also converges.

In part (b) students were asked to use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely. A correct response should use correct notation to show that $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left| \frac{(-1)^n}{2e^n + 3} \right|}$ is finite and positive and reference the convergence of $\sum_{n=0}^{\infty} \frac{1}{e^n}$ determined in part (a).

In part (c) students were asked to determine the radius of convergence of the Maclaurin series for g. A correct response should use the ratio test to determine the radius of convergence is $R = e$.

In part (d) students were asked to use the alternating series error bound to determine an upper bound on the error when the first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. A correct response should indicate that the approximate error is bounded by the absolute value of the third term of the series, $\frac{1}{2e^2 + 3}$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

In part (a) many responses were able to identify the three conditions necessary for applying the integral test and write the improper integral needed to apply the integral test in this situation. Most responses were able to find an antiderivative of $f(x) = e^{-x}$ and evaluate the necessary limit, thereby demonstrating that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. However, correct notation was frequently not used in evaluating the improper integral, and some responses used inappropriate notation such as $e^{-\infty}$. Some responses attempted to apply the Fundamental Theorem of Calculus in order to evaluate the improper integral.

In part (b) some responses were able to set up a correct limit of a quotient of the two given series, including absolute values either explicitly or implicitly. These responses typically evaluated the limit and observed that it was positive and finite and correctly concluded that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely. Many responses, however, failed to consider the absolute values and therefore incorrectly evaluated the limit. Many responses incorrectly compared the evaluated limit to the value 1. Several responses also attempted to apply the comparison test rather than the limit comparison test.

In part (c) many responses were able to set up a correct ratio in order to apply the ratio test. They then evaluated the limit of this ratio and wrote an inequality indicating that the limit was less than 1. Some responses then correctly reported the radius of convergence, but many responses instead reported an interval of convergence (without endpoint analysis). The most common error in part (c) was using an incorrect ratio initially, and addition, many responses neglected to use absolute values at any time during their evaluation of the limit.

In part (d) responses were generally able to identify the third nonzero term of the alternating series as an upper bound for the error. Some responses instead used the fourth nonzero term, and some attempted to use a list of terms.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses failed to make a direct connection between the conditions necessary for an integral test and the function $f(x) = e^{-x}$, and/or incorrectly evaluated the improper integral.	• e^{-x} is positive, decreasing, and continuous. $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left(-e^{-x}\Big\vert_0^b\right)$ $= \lim_{b\to\infty}\left(-e^{-b} + e^0\right) = 1$
• In part (b) many responses failed to use absolute values on the limit and/or failed to compare the evaluated limit to a positive, finite value.	• $\lim_{n\to\infty}\dfrac{\frac{1}{e^n}}{\left\lvert\frac{(-1)^n}{2e^n+3}\right\rvert} = \lim_{n\to\infty}\dfrac{2e^n+3}{e^n} = 2$ This limit exists and is positive, therefore $\sum_{n=0}^{\infty}\left\lvert\frac{(-1)^n}{2e^n+3}\right\rvert$ converges by the limit comparison test and the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2e^n+3}$ converges absolutely.
• In part (c) many responses failed to use absolute values, and some responses confused the interval of convergence with the radius of convergence.	• $\left\lvert\dfrac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3}\cdot\dfrac{2e^n+3}{(-1)^n x^n}\right\rvert = \dfrac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert$ $\lim_{n\to\infty}\dfrac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert = \dfrac{1}{e}\lvert x\rvert$ $\frac{1}{e}\lvert x\rvert < 1 \Rightarrow \lvert x\rvert < e$ The radius of convergence is e.

• In part (d) some responses used the fourth term of the series as the error bound or provided a nonnumerical error bound such as $\left\lvert \dfrac{(-1)^n}{2e^n + 3} \right\rvert$.	• By the alternating series error bound, Error $\le \left\lvert \dfrac{(-1)^2}{2e^2 + 3} \right\rvert = \dfrac{1}{2e^2 + 3}$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

Teachers should provide as much practice as possible asking students to write calculus-based explanations, requiring specific identification of *which* function, graph, series, or term is being referenced. Teachers should discourage the use of any short-hand notation for conceptual ideas, such as e^{∞}. Teachers should be diligent in using absolute values whenever appropriate and should require students to use them as well, never omitting them without explanation.

When teaching series, teachers should stress both the difference between and the correct use of terms such as radius of convergence and interval of convergence, and make sure students are careful to read the question and answer the question asked.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The *AP Calculus AB and BC Course and Exam Description* (*CED*), effective Fall 2020, includes instructional resources for AP Calculus teachers to develop students' broader skills. Although convergence tests are specified in this question, it is important to be able to identify an appropriate test (skill 3.B). Please see page 217 of the *CED* for examples of key questions and instructional strategies designed to develop skill 3.B. A table of representative instructional strategies, including definitions and explanations of each, is included on pages 203–213 of the *CED*. The strategy "Graphic Organizer," for example, provides students a visual system for organizing features of the multiple tests learned, including the conditions that must be met to perform the integral test (skill 3.C) and differences between the direct comparison test and the limit comparison test.
- AP Classroom provides videos focused on the content and skills needed to answer this question.
    - The videos for topic 10.4 provide a thorough treatment of confirming the conditions that must be met to implement the integral test, along with correct notation for and evaluation of improper integrals. For additional review of improper integrals, consider viewing the videos for topic 6.13.
    - The videos for topic 10.6 develop understanding of the conceptual underpinnings to the direct and limit comparison tests and provide worked examples and opportunities for student practice.
    - The videos for topic 10.13 develop the concepts and skills necessary to determine a radius or interval of convergence of a power series. Please note that in the case of this question, responses may have answered the wrong question, offering an interval, rather than a radius of convergence. It is always important to identify the question to be answered or problem to be solved (skill 1.A).
    - The video for topic 10.10 provides conceptual underpinnings, information about the alternating series error bound, and examples to develop mastery of the related content and skills.
- AP Classroom also provides topic questions for formative assessment and access to the question bank, which is a searchable database of past AP Questions on these topics.
- The Online Teacher Community features many resources shared by other AP Calculus teachers. For example, to locate resources to give your students practice with convergence tests, try entering the keyword "convergence tests" in the search bar, then selecting the drop-down menu for "Resource Library." When you filter for "Classroom-Ready Materials," you may find worksheets, data sets, practice questions, and guided notes, among other resources.