2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1)$$
$$= 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2$$
$$= 3 + x + x^2.$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $h(1) = \int_0^1 f(t)\,dt$

$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$$
$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$$
$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

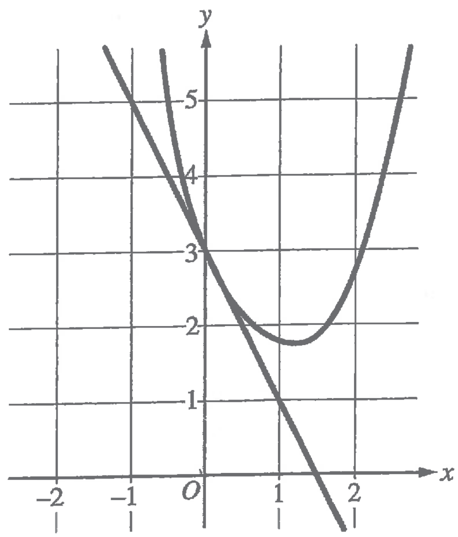
$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$\sum_{n=0}^{3} \frac{f^{n}(0)x^n}{n!}$$

$$3 - 2x + \frac{3x^2}{2!} - \frac{23x^3}{2\cdot 3!}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$\left(1+x+\frac{x^2}{2}\right)\left(3-2x+\frac{3x^2}{2}-\frac{23x^3}{12}\right)$$

$$3-2x+\frac{3x^2}{2}+3x-2x^2+\frac{3x^2}{2}+\ldots$$

$$3+x+x^2\left(\frac{3}{2}-2+\frac{3}{2}\right)$$

$$3+x+x^2$$

Continue question 6 on page 2:

NO CALCULATOR ALLOWED

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(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$$h(x) = 3x - x^2 + \frac{x^3}{2} - \frac{23x^4}{8 \cdot 3!}$$

$$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{8 \cdot 3!}$$

$$= 2 + \frac{1}{2} - \frac{23}{48}$$

$$= \frac{96 + 24 - 23}{48}$$

$$= \frac{97}{48}$$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$$T_4(x) = \int_0^x \frac{54t^4}{4!}\,dt = \frac{54x^5}{5!}$$

$$T_4(1) = \frac{54}{5!} = \frac{54}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{54}{20 \cdot 6} = \frac{54}{120} = \frac{27}{60} = \frac{9}{20}$$

error bound = fourth term = $\frac{9}{20}$ = 0.45

$$\frac{9}{20} \le 0.45$$

error $\le$ 0.45

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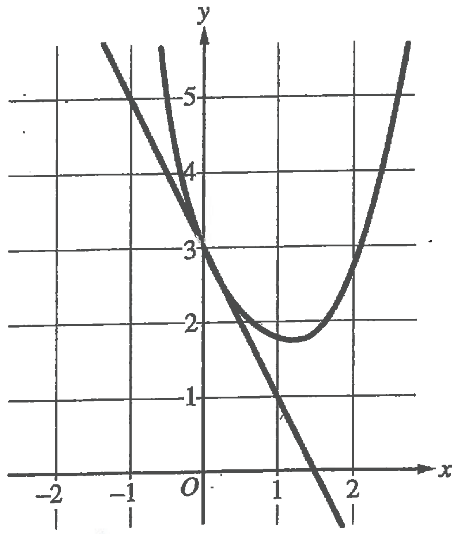

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$P_3 = \frac{3x^0}{0!} + \frac{-2x^1}{1!} + \frac{3x^2}{2!} - \frac{23x^3}{2\cdot 3!}$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$1 + x + \frac{x^2}{2}$$

$$3 - 2x + \frac{3}{2}x^2$$
$$\times\ 1 + x + \frac{x^2}{2}$$
$$3 - 2x + \frac{3}{2}x^2$$
$$3x - 2x^2$$
$$\frac{3}{2}x^2$$

$$\boxed{P_2 = 3 + x + x^2}$$

Continue question 6 on page 21.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(x) = \int_0^x 3 - 2t + \frac{3}{2}t^2 - \frac{23}{36}t^3\,dt$

$= 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{36}x^4$

$h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{36}$

$= \frac{5}{2} - \frac{23}{36}$

$= \frac{67}{36}$

$\begin{array}{r} 18 \\ \times\ 5 \\ \hline 90 \\ -\ 23 \\ \hline 67 \end{array}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$4! = 24 = 4 \cdot 3 \cdot 2$

$\frac{f^{(4)}(0)\,x^4}{4!} = \frac{54\,x^4}{4!} = \frac{27x^4}{12}$

$\frac{\frac{27}{5}x^5}{12} = \frac{27x^5}{60}$

$\frac{27}{60} \leq 0.45$

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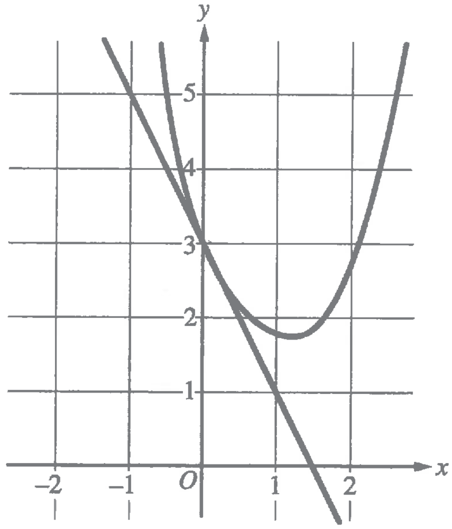

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \frac{f'''(0)(x)^3}{3!} = T_3(x)$$

$$3 + \left(\frac{5-3}{-1-0}\right)x + \frac{3x^2}{2} + -\frac{23}{2}\cdot\frac{x^3}{6} = T_3(x)$$

$$\boxed{3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3 = T_3(x)}$$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$M(x) = 1 + x + \frac{x^2}{2}$$

$$T_2(x) = f(0)\left(1 + x + \frac{x^2}{2}\right) = 3\left(1 + x + \frac{x^2}{2}\right)$$

$$\boxed{T_2(x) = 3 + 3x + \frac{3}{2}x^2}$$

Continue question 6 on page 21.

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(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$h(1) = \int_0^1 f(t)\,dt$

$h(1) \approx 3 - 2(1) + \frac{3}{2}(1)^2 - \frac{23}{12}(1)^3$

$h(1) \approx 1 + \frac{3}{2} - \frac{23}{12}$

$h(1) \approx \frac{30}{12} - \frac{23}{12}$

$\boxed{h(1) \approx \frac{7}{12}}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$\frac{54x^4}{24} \leq 0.45$

AP® CALCULUS BC
2019 SCORING COMMENTARY

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function f is presented that has derivatives of all orders for all real numbers x. A figure showing a portion of the graph of f and a line tangent to the graph of f at $x = 0$ is given, as is a table showing values for $f^{(2)}(0)$, $f^{(3)}(0)$, and $f^{(4)}(0)$.

In part (a) students were asked to write the third-degree Taylor polynomial for f about $x = 0$. A response should demonstrate that terms of the Taylor polynomial have the form $\frac{f^{(n)}(0)}{n!}x^n$, determine $f(0)$ and $f'(0)$ from the given graph, and find values for $f''(0) = f^{(2)}(0)$ and $f^{(3)}(0)$ in the table to construct the requested Taylor polynomial.

In part (b) students were asked to write the first three nonzero terms of the Maclaurin series for e^x and to provide the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$. A response should state that the Maclaurin series for e^x starts with the terms $1 + x + \frac{1}{2!}x^2 + \cdots$ and then form the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ using the terms of degree at most 2 in the product $\left(1 + x + \frac{1}{2!}x^2\right) \cdot T_3(x)$, where $T_3(x)$ is the Taylor polynomial found in part (a).

In part (c) students were asked to use the Taylor polynomial found in part (a) to approximate $h(1)$, where $h(x) = \int_0^x f(t)\,dt$. A response should demonstrate that $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 T_3(t)\,dt$ where $T_3(x)$ is the Taylor polynomial found in part (a). $\int_0^1 T_3(t)\,dt$ should be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (d) it is given that the Maclaurin series for h converges to $h(x)$ everywhere and that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Students were asked to use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45. A response should demonstrate that the error in the approximation is bounded by the magnitude of the first omitted term of the series for $h(1)$. This term is found by integrating the fourth-degree term of the Taylor series for f about $x = 0$ across the interval $[0, 1]$. Computing this term demonstrates the desired error bound.

For part (a) see LO LIM-8.A/EK LIM-8.A.1. For part (b) see LO LIM-8.F/EK LIM-8.F.2, LO LIM-8.G/EK LIM.8.G.1. For part (c) see LO LIM-8.G/EK LIM-8.G.1. For part (d) see LO LIM-8.C/EK LIM-8.C.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

Question 6 (continued)

Sample: 6A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the response earned the first point in line 2 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 2 for the remaining terms, $+\frac{3x^2}{2!} - \frac{23x^3}{2 \cdot 3!}$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 for the first three nonzero terms of the Maclaurin series for e^x, $1 + x + \frac{x^2}{2}$. The response earned the second point in line 5 with the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$, $3 + x + x^2$, with supporting work. In part (c) the response earned the first point in line 1 with the correct antiderivative $3x - x^2 + \frac{x^3}{2} - \frac{23x^4}{8 \cdot 3!}$. Numerical simplification is not required. The response would have earned the second point in line 2 with the evaluation $h(1) = 3 - 1 + \frac{1}{2} - \frac{23}{8 \cdot 3!}$ and no numerical simplification. The response simplifies to $\frac{97}{48}$ in line 5 and earned the second point. In part (d) the response earned the first point in line 1 with $\int_0^x \frac{54t^4}{4!}\,dt$. The response would have earned the second point in line 2 with $\frac{54}{5!}$ without simplification. The second point was earned with a correct simplification of $\frac{9}{20}$ at the end of line 2. The response earned the third point in lines 3 and 5 with "$\frac{9}{20} = 0.45$" and "error ≤ 0.45."

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the first point in line 2 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 2 for the remaining terms, $+\frac{3}{2}x^2 - \frac{23}{12}x^3$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 on the left for the first three nonzero terms of the Maclaurin series for e^x, $1 + x + \frac{x^2}{2}$. The response earned the second point in the last line with the boxed second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$, $P_2 = 3 + x + x^2$, with supporting work. In part (c) the response did not earn the first point because the response has a copy error in the expression for $f(x)$ in the fourth term of the integrand, $-\frac{23}{36}t^3$, in line 1. The missing parentheses in the integrand do not impact earning the point. The response also has an antidifferentiation error in line 2 in the fourth-degree term (a missing factor of 4 in the denominator). Because the response has two errors, the response is not eligible for the second point. In part (d) the response would have earned the first point in line 2 with $\frac{\frac{27}{5}x^5}{12}$. The

Question 6 (continued)

first point was earned with the correct simplification, $\frac{27x^5}{60}$. The response earned the second point in line 3 with $\frac{27}{60}$. The response did not earn the third point because there is no reference to the error being bounded.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the response earned the first point in line 3 for the first two terms, $3 - 2x$, of the third-degree Taylor polynomial for f. The response earned the second point in line 3 for the remaining terms, $+\frac{3}{2}x^2 - \frac{23}{12}x^3$, of the third-degree Taylor polynomial for f. In part (b) the response earned the first point in line 1 for the first three nonzero terms of the Maclaurin series for e^x, $M(x) = 1 + x + \frac{x^2}{2}$. The response did not earn the second point because the Taylor polynomial for $e^x f(x)$, $3\left(1 + x + \frac{x^2}{2}\right)$, in line 2 is incorrect. In part (c) the response did not earn the first point because the third-degree Taylor polynomial found in part (a) is not antidifferentiated. Because the response does not antidifferentiate the third-degree Taylor polynomial from part (a), the response is not eligible for the second point. In part (d) the response did not earn the first point because there is no attempt to antidifferentiate the fourth-degree term of f. Without an attempt to antidifferentiate the fourth-degree term of f, the response is not eligible to earn either the second or the third point.