2019

AP® | CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$
$$= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1}$$
$$= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b \to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b \to 1^-} \left(-\frac{1}{x-1} \Bigg|_{x=0}^{x=b} \right) + \lim_{b \to 1^+} \left(-\frac{1}{x-1} \Bigg|_{x=b}^{x=2} \right)$$
$$= \lim_{b \to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\displaystyle\lim_{b \to 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

5A
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5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$f(x) = (x^2-2x+k)^{-1}$

$f'(x) = -(x^2-2x+k)^{-2}(2x-2)$

$= -\dfrac{2x-2}{(x^2-2x+k)^2}$

$f'(0) = -\dfrac{-2}{k^2} = 6$

$6k^2 = 2$

$\boxed{k = \dfrac{1}{\sqrt{3}}}$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$\dfrac{A}{(x-4)} + \dfrac{B}{(x+2)} = \dfrac{1}{(x-4)(x+2)}$

$A(x+2) + B(x-4) = 1$

$6A = 1 \quad A + B = 0$

$A = \frac{1}{6} \quad B = -\frac{1}{6}$

$\int_0^1 \dfrac{1}{x^2-2x-8}\,dx$

$\int_0^1 \dfrac{\frac{1}{6}}{x-4} - \dfrac{\frac{1}{6}}{x+2}\,dx =$

$\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|\Big|_0^1 =$

$\frac{1}{6}\ln\left|\dfrac{x-4}{x+2}\right|\Bigg|_0^1 = \frac{1}{6}\ln\left|\dfrac{-3}{3}\right| - \frac{1}{6}\ln\left|\dfrac{-4}{2}\right| = \frac{1}{6}(0 - \ln 2) = \boxed{-\frac{1}{6}\ln 2}$

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NO CALCULATOR ALLOWED

5A
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(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2-2x+1}\,dx$$

$x^2-2x+1=0$
$(x-1)^2=0$
$x=1$

$$\lim_{R\to1^-}\int_0^R \frac{1}{(x-1)^2}\,dx + \lim_{R\to1^+}\int_R^2 \frac{1}{(x-1)^2}\,dx$$

$$\lim_{R\to1^-} -\frac{1}{x-1}\Big|_0^R + \lim_{R\to1^+} -\frac{1}{x-1}\Big|_R^2$$

$$\lim_{R\to1^-}\left(-\frac{1}{R-1}+\frac{1}{-1}\right)+\lim_{R\to1^+}\left(-\frac{1}{2-1}+\frac{1}{R-1}\right)$$

$\infty - 1 - 1 + \infty$

$\boxed{\text{The integral diverges}}$

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5B
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5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2-2x+k)(0) - 1(2x-2)}{(x^2-2x+k)^2}$$

$$f'(x) = \frac{-2x+2}{(x^2-2x+k)^2} \qquad f'(0) = 6$$

$$\frac{-2(0)+2}{(0^2-2(0)+k)^2} = 6$$

$$\frac{2}{k^2} = 6$$

$$2 = 6k^2$$

$$\boxed{k = \pm\sqrt{\frac{1}{3}}}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\, dx$.

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-4)$$

$$1 = (A+B)x + 2A - 4B$$

$$(A+B=0)\,{-2}$$

$$2A - 4B = 1$$

$$-2A - 2B = 0 \qquad A - \frac{1}{6} = 0$$

$$-6B = 1 \qquad A = \frac{1}{6}$$

$$B = -\frac{1}{6} \qquad \frac{1}{(x-4)(x+2)} = \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}$$

$$\int_0^1 \frac{1}{x^2-2x-8}\, dx = \int_0^1 \frac{1}{(x-4)(x+2)}$$

$$= \frac{1}{6}\int_0^1 \frac{1}{x-4}\, dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\, dx$$

$$= \frac{1}{6}\left[\ln|x-4|\Big|_0^1 - \ln|x+2|\Big|_0^1\right]$$

$$= \frac{1}{6}\left[\ln 3 - \ln 4 - \ln 3 + \ln 2\right]$$

$$= \frac{1}{6}(\ln 2 - \ln 4)$$

$$= \boxed{\frac{1}{6}\ln\frac{1}{2}}$$

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NO CALCULATOR ALLOWED

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(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$x^2 - 2x + 1 = (x-1)^2$

$\int_0^2 \frac{1}{x^2-2x+1}\,dx$

$\int_0^2 \frac{1}{(x-1)^2}\,dx$ let $u = x-1$

$\int_{-1}^1 \frac{1}{u^2}\,du$ $\frac{du}{dx} = 1$

$du = dx$

$-\frac{1}{u}\Big|_{-1}^{1}$

$-\frac{1}{1} - \frac{1}{1}$

$\boxed{-2}$

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5C
1&2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2-2x+k) - (2x-2)}{(x^2-2x+k)^2} = \frac{x^2-4x+2+k}{(x^2-2x+k)^2}$$

$0^2 - 4(0) + 2 + k = 6 \quad 2+k=6$

$k=4$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$$\int_0^1 \frac{1}{x^2-2x-8}\,dx = \int_0^1 \frac{1}{(x-4)(x+2)}\,dx = \int_0^1 \frac{A}{x-4} + \frac{B}{x+2}\,dx = \frac{1}{6}\int_0^1 \frac{1}{x-4}\,dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\,dx$$

$1 = Ax + 2A + Bx - 4B$

$A + B = 0 \quad A = -B$

$2A - 4B = 1$

$-6B = 1$

$B = \frac{-1}{6} \quad A = \frac{1}{6}$

$\frac{1}{6}\left(\ln\frac{-1}{3} - \ln\frac{-1}{4}\right) - \frac{1}{6}\left(\ln\frac{1}{3} - \ln\frac{1}{2}\right)$

$\frac{1}{6}\ln\frac{4}{3} - \frac{1}{6}\ln\frac{2}{3}$

$\frac{1}{6}\ln\frac{4}{2} = \frac{\ln 2}{6}$

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NO CALCULATOR ALLOWED

5C
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(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \frac{-1}{x-1}\Big]_0^2 = \frac{-1}{1} - 1 = -2$$

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Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem deals with a family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

In part (a) students were asked to find the positive value of k such that the slope of the line tangent to the graph of f at $x = 0$ equals 6. A response should demonstrate differentiation rules to find $f'(x)$ and then identify $f'(0)$ as the slope of the line tangent to the graph of f at $x = 0$, so the k can be found by solving $f'(0) = 6$.

In part (b) students were asked to evaluate $\int_0^1 f(x)\,dx$ in the case where $k = -8$. A response should demonstrate that, with $k = -8$, $f(x)$ can be expressed using partial fractions as $f(x) = \dfrac{1}{x^2 - 2x - 8} = \dfrac{\frac{1}{6}}{x - 4} - \dfrac{\frac{1}{6}}{x + 2}$. Then $\int_0^1 f(x)\,dx$ can be evaluated using antidifferentiation and the Fundamental Theorem of Calculus.

In part (c) students were asked to evaluate $\int_0^2 f(x)\,dx$ or show that it diverges, in the case where $k = 1$. A response should note that $x^2 - 2x + 1 = (x - 1)^2$, so the graph of f has a vertical asymptote at $x = 1$ and $\int_0^2 f(x)\,dx$ is an improper integral. Thus $\int_0^2 f(x)\,dx$ is the sum $\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$, providing each of the summands converges. Expressing either summand as a one-sided limit of a proper integral, a response should demonstrate that the summand diverges and conclude that $\int_0^2 f(x)\,dx$ diverges.

For part (a) see LO FUN-3.C/EK FUN-3.C.1, LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-6.F.b/EK FUN-6.F.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO LIM-6.A/EK LIM-6.A.1, LO FUN-6.C/EK FUN-6.C.2, LO LIM-6.A/EK LIM-6.A.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, and Practice 4: Communication and Notation.

Sample: 5A
Score: 9

The response earned 9 points: 3 points in part (a), 3 points in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 2 with the term $\left(x^2 - 2x + k\right)^{-2}$ in the presented derivative expression $f'(x) = -\left(x^2 - 2x + k\right)^{-2}(2x - 2)$. The second point was also earned in line 2 with this presented expression for the derivative $f'(x)$. The third point was earned in line 6 with the declaration that $k = \dfrac{1}{\sqrt{3}}$ and the consistent work leading to this value. In part (b) the response earned the first point in line 1 on the right with the equation $\dfrac{A}{(x - 4)} + \dfrac{B}{(x + 2)} = \dfrac{1}{(x - 4)(x + 2)}$ and the declaration in line 4 on the right that $A = \dfrac{1}{6}$ and $B = -\dfrac{1}{6}$. The response earned the second point in line 3 on the left with the antiderivatives $\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|$. The response would have earned the third point in line 4 on the left with the expression $\frac{1}{6}\ln\left|\frac{-3}{3}\right| - \frac{1}{6}\ln\left|\frac{-4}{2}\right|$ and no numerical simplification. The response simplifies the expression correctly, and the third point was earned with $-\frac{1}{6}\ln 2$. In part (c) the response earned the first point in line 2 with the expression $\lim\limits_{R\to 1^-}\int_0^R \frac{1}{(x-1)^2}\,dx + \lim\limits_{R\to 1^+}\int_R^2 \frac{1}{(x-1)^2}\,dx$. The response earned the second point in line 3 with the antiderivative $-\frac{1}{x-1}$ in the expression $\lim\limits_{R\to 1^-} -\frac{1}{x-1}\Big|_0^R + \lim\limits_{R\to 1^+} -\frac{1}{x-1}\Big|_R^2$. The response earned the third point in line 6 with the boxed conclusion "[t]he integral diverges" and with the correct work that includes both one-sided limits evaluated in line 5 as $\infty - 1 - 1 + \infty$.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the response earned the first point in line 1 with the denominator $\left(x^2 - 2x + k\right)^2$ in the presented derivative expression $f'(x) = \frac{\left(x^2 - 2x + k\right)(0) - 1(2x-2)}{\left(x^2 - 2x + k\right)^2}$. The second point was also earned in line 1 with this presented expression for the derivative $f'(x)$. The response did not earn the third point, as the value $k = \pm\sqrt{\frac{1}{3}}$ in line 6 does not utilize the given information that $k > 0$. In part (b) the response earned the first point in line 1 on the right with the equation $\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$ and the declaration in lines 7 and 8 on the right that $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. The response earned the second point in line 3 on the left with the antiderivatives $\frac{1}{6}\left[\ln|x-4|\Big|_0^1 - \ln|x+2|\Big|_0^1\right]$. The response would have earned the third point in line 4 on the left with the expression $\frac{1}{6}[\ln 3 - \ln 4 - \ln 3 + \ln 2]$ and no numerical simplification. The response simplifies this expression correctly to the boxed $\frac{1}{6}\ln\frac{1}{2}$ in line 6 on the left, and the third point was earned with this final expression. In part (c) the response did not earn the first point because there is no indication that the integral $\int_0^2 \frac{1}{(x-1)^2}\,dx$ is improper. The response earned the second point by letting $u = x - 1$ in line 2 and with antiderivative $-\frac{1}{u}$ in line 4. The response did not earn the third point because there is no conclusion of divergence.

Question 5 (continued)

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the response earned the first point in line 1 with the correct denominator $\left(x^2 - 2x + k\right)^2$ in the presented derivative expression $f'(x) = \dfrac{\left(x^2 - 2x + k\right) - (2x - 2)}{\left(x^2 - 2x + k\right)^2}$. The response did not earn the second point because the presented derivative for $f'(x)$ in line 1 is incorrect. The response did not earn the third point because the circled answer $k = 4$ on the right is incorrect. In part (b) the response earned the first point in line 1 with the equation $\int_0^1 \dfrac{1}{(x-4)(x+2)}\,dx = \int_0^1 \dfrac{A}{x-4} + \dfrac{B}{x+2}\,dx$ and with the declaration in the last line on the right that $B = \dfrac{-1}{6}$ and $A = \dfrac{1}{6}$. The missing parentheses in the integrand on the right side of the equation do not impact earning the point. The response did not earn the second point because there is no antiderivative presented. The response did not earn the third point due to the undefined expression $\dfrac{1}{6}\left(\ln\dfrac{-1}{3} - \ln\dfrac{-1}{4}\right) - \dfrac{1}{6}\left(\ln\dfrac{1}{3} - \ln\dfrac{1}{2}\right)$ in line 3 on the left. Note that even if the circled answer $\dfrac{\ln 2}{6}$ had been correct, this response would not have been eligible for the third point because of the undefined, incorrect expression. In part (c) the response did not earn the first point because there is no indication that the integral $\int_0^2 \dfrac{1}{(x-1)^2}\,dx$ is improper. The response earned the second point with the antiderivative $\dfrac{-1}{x-1}$. The response did not earn the third point because there is no conclusion of divergence.