2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $$\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

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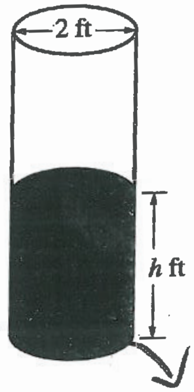


4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$h=4$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{4} = \frac{-1}{5}$

$\frac{dV}{dt} = ?$

$r=1$

$\frac{dr}{dt} = 0$

$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right]$

$\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$

$\frac{dV}{dt} = \frac{-\pi}{5}\ ft^3/sec$

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$h=3$

$\frac{d^2h}{dt^2}=?$

$\frac{-1}{10}h^{1/2}$

$$\frac{d^2h}{dt^2}=\frac{-1}{20}h^{-1/2}\frac{dh}{dt}=\frac{-1}{20\sqrt{3}}\left(\frac{-1}{10}\sqrt{3}\right)=\frac{1}{200}$$

The rate of change of height is increasing since $\frac{d^2h}{dt^2}$ at $h=3$ is positive.

$$\frac{1}{200}$$

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$$\int h^{-1/2}dh=\int\frac{-1}{10}dt$$

$$2h^{1/2}=\frac{-1}{10}t+C$$

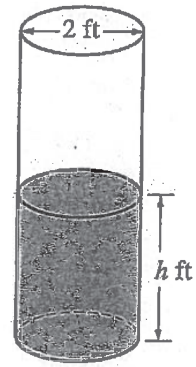
$$h^{1/2}=\frac{-1}{20}t+C$$

$$h=\left(\frac{-1}{20}t+C\right)^2$$

$$5=\left(\frac{-1}{20}(0)+C\right)^2$$

$$C=\sqrt{5}$$

$$h=\left(\frac{-1}{20}t+\sqrt{5}\right)^2$$



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(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

for $h = 4$:

$\frac{dV}{dt} = \pi(1)^2 \cdot \left(-\frac{1}{10}\sqrt{4}\right) = \boxed{-\frac{\pi}{5} \text{ feet}^3/\text{s}}$

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

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(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

at $h=3$, $\frac{dh}{dt} = -\frac{1}{10}\sqrt{3}$, which is negative, so the amount of water is decreasing

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(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{1}{\sqrt{h}}dh = -\frac{1}{10}dt$

$\int \frac{1}{\sqrt{h}}dh = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{t}{10} + C$

$2\sqrt{5} = -\frac{0}{10} + c$

$c = 2\sqrt{5}$

$2\sqrt{h} = -\frac{t}{10} + 2\sqrt{5}$

$\sqrt{h} = -\frac{t}{20} + \sqrt{5}$

$\boxed{h = \left(-\frac{t}{20} + \sqrt{5}\right)^2}$

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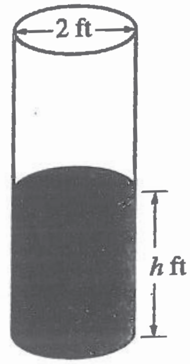

4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$\frac{dv}{dt} = ?$ $h=4$

$V=\pi r^2 h$

$r=1$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{h}$

$V=\pi r^2 h$

$\frac{dV}{dt} = 2\pi r \frac{dh}{dt}$

$\frac{dv}{dt} = 2\pi(1)\left(-\frac{1}{10}\sqrt{4}\right)$

$\frac{dv}{dt} = 2\pi \cdot -\frac{2}{10} = -\frac{2\pi}{5}$ ft^3 per second

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$\frac{dh}{dt}$

$r(t) = \frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$

$r'(3) = -\frac{1}{10} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} = -\frac{1}{20\sqrt{3}} < 0$

When the height of the water is 3 feet the rate of change of height of the water is decreasing because $r'(t) < 0$.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$

$2h^{1/2} + c_1 = -\frac{1}{10} t + c_2$

$h = \sqrt{-\frac{1}{20} t + c}$

$5 = \sqrt{-\frac{1}{20}(0) + c}$

$c = 25$

$h = \sqrt{-\frac{1}{20} t + 25}$

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context for this problem is a cylindrical barrel with a diameter of 2 feet that contains collected rainwater, some of which drains out through a valve in the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet, and t is measured in seconds.

In part (a) students were asked to find the rate of change of the volume of water in the barrel with respect to time when $h = 4$ feet. A response should use the geometric relationship between the volume V of water in the barrel and height h and incorporate the given expression for $\frac{dh}{dt}$.

In part (b) students were asked to determine whether the rate of change of the height of water in the barrel is increasing or decreasing when $h = 3$ feet. A response should demonstrate facility with the chain rule to differentiate $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ with respect to time to obtain $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$. Because $\frac{d^2h}{dt^2} > 0$, a response should conclude that the rate of change of the height of the water in the barrel is increasing.

In part (c) students were given that the height of the water is 5 feet at time $t = 0$ and then asked to use the technique of separation of variables to find an expression for h in terms of t. A response should demonstrate the application of separation of variables to solve the differential equation $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ for h and then incorporate the initial condition that $h(0) = 5$ to find the particular solution $h(t)$ to the differential equation.

For part (a) see LO CHA-3.D/EK CHA-3.D.1, LO CHA-3.E/EK CHA-3.E.1. For part (b) see LO FUN-4.E/EK FUN-4.E.2. For part (c) see LO FUN-7.D/EK FUN-7.D.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-7.E/EK FUN-7.E.1. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 4A
Score: 9

The response earned 9 points: 2 points in part (a), 3 points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to t, $\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right]$, in line 1 on the right. The second point would have been earned with $\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$ in line 2 on the right. Although numerical simplification is not required, the response simplifies the expression in line 3 on the right and adds units to produce $-\frac{\pi}{5}$ ft^3/sec. Thus the second point was earned. In part (b) the response presents a correct second derivative of h with respect to t, $\frac{d^2h}{dt^2} = -\frac{1}{20}h^{-\frac{1}{2}}\frac{dh}{dt}$, in line 1 on the right and earned both the first and second points. The response earned the

Question 4 (continued)

third point in lines 2, 3, and 4 on the right with "[t]he rate of change of height is increasing since $\frac{d^2h}{dt^2}$ at $h = 3$ is positive." In part (c) the response earned the first point with a correct separation of variables $\int h^{-\frac{1}{2}}\, dh = \int \frac{-1}{10}\, dt$ in line 1. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t,$ are presented in line 2, and the response earned the second point. In lines 2 and 5, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(\frac{-1}{20}t + \sqrt{5}\right)^2$ in line 7.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt},$ while handling r as a constant. The second point would have been earned with $\frac{dV}{dt} = \pi(1)^2\left(-\frac{1}{10}\sqrt{4}\right)$ in line 3. Although numerical simplification is not required, the response simplifies the expression in line 3 and adds units to produce $-\frac{\pi}{5}$ feet^3/s. Thus the second point was earned. In part (b) the response does not include the derivative of $\frac{dh}{dt},$ so the first and second points were not earned. Because there is no second derivative, the response is not eligible for the third point. In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$ in line 1 on the left. The correct antiderivatives, $2\sqrt{h}$ and $-\frac{t}{10},$ are presented in line 3 on the left, and the response earned the second point. In lines 3 and 4 on the left, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(-\frac{t}{20} + \sqrt{5}\right)^2$ in the box in line 3 on the right.

Sample: 4C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the response presents an incorrect expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = 2\pi r \frac{dh}{dt},$ in line 3 on the left. The first point was not earned, and this error makes the response not eligible for the second point. In part (b) the response defines $r(t) = \frac{dh}{dt}$ in line 1. The response earned the first point with $r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$ in line 2. The expression is identified as the second derivative of h with respect to $t;$ however, $r'(t)$ does not

Question 4 (continued)

include a factor of $\frac{dh}{dt}$. Thus the second point was not earned, and the response is not eligible for the third point.

In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\,dh = -\frac{1}{10}\,dt$ in line 2 on the left. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t,$ are presented in line 3 on the left, and the response earned the second point. In line 3 on the left, the response includes constants of integration. The response incorrectly solves for h in terms of t in line 4 before using the initial condition $h(0) = 5$ in line 5. The resulting expression $h = \sqrt{-\frac{1}{20}t + C}$ in line 4 on the left is incorrect. Thus the response is not eligible for the third and fourth points.