2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 3

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Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

NO CALCULATOR ALLOWED

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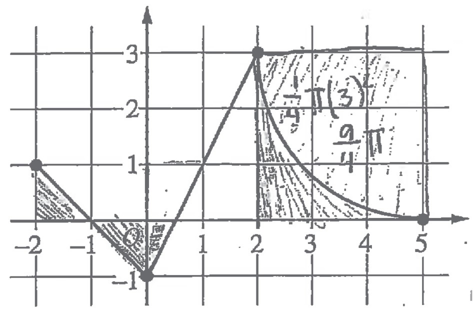

$3 \cdot \frac{3}{2} = \frac{9}{2} \cdot \frac{1}{2} = \frac{9}{4}$

Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$$

$$\int_{-6}^{-2} f(x)\,dx = 7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$\int_3^5 2f'(x) + 4\,dx = \int_3^5 2(f'(x) + 2)\,dx$$

$$= 2\int_3^5 f'(x) + 2\,dx$$

$$= 2 \cdot \left[f(x) + 2x\right]_3^5$$

$$= 2 \cdot \left[\left[f(5) + 2(5)\right] - \left[f(3) + 2(3)\right]\right]$$

$$= 2 \cdot \left[(0 + 10) - \left((3 - \sqrt{5}) + 6\right)\right]$$

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(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

Since $f(x)$ is continuous on $-2 \le x \le 5$, then $g(x) = \int_{-2}^{x} f(t)\,dt$ is also continuous on $-2 \le x \le 5$. Therefore, by the EVT, $g(x)$ has an absolute max. on $-2 \le x \le 5$.

$g'(x) = f(x)$

$g'(x) = f(x) = 0$

$x = -1,\ x = \frac{1}{2},\ x = 5$

Candidates for abs. max: $x = -2, -1, \frac{1}{2}, 5$

$x = -2: \int_{-2}^{-2} f(t)\,dt = 0$

$x = -1: \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$

$x = \frac{1}{2}: \int_{-2}^{1/2} f(t)\,dt = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} = -\frac{1}{4}$

$x = 5: \int_{-2}^{-5} f(t)\,dt = -\frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right) = 2 + 9 - \frac{9\pi}{4} = 11 - \frac{9\pi}{4}$

The absolute maximum value of g is $11 - \frac{9\pi}{4}$ and the absolute maximum occurs at $x = 5$.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$\lim_{x \to 1} 10^x - 3f'(x) = 10 - 3f'(1) = 10 - 3(2) = 4$

$\lim_{x \to 1} f(x) - \arctan x = 1 - \pi/4$

$\arctan(1) = \pi/4$

$\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{4}{1 - \pi/4}$

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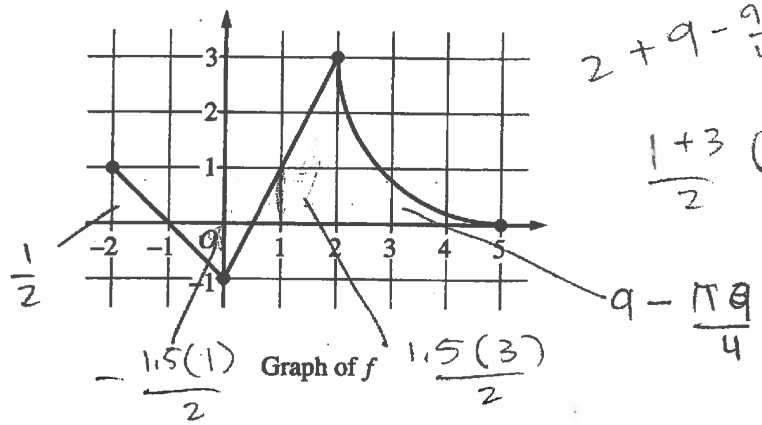
$2+9-\frac{9\pi}{4}$

$\frac{1+3}{2}(1)$



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)dx$$

$$\int_{-6}^{5} f(x)dx - \int_{-2}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx$$

$$7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right) = \boxed{-2 + \frac{9\pi}{4}}$$

$\frac{8}{4} + 9 - \frac{9\pi}{4}$

$11 - \frac{9\pi}{4}$

$7 - 11 + \frac{9\pi}{4} =$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$2\int_{3}^{5} f'(x)dx + \int_{3}^{5} 4\,dx$$

$$2\left[f(x)\right]_{3}^{5} + \left[4x\right]_{3}^{5}$$

$$2(f(5) - f(3)) + 20 - 12$$

$$2(0 - 3 + \sqrt{5}) + 8$$

$2(-3+\sqrt{5}) + 8$

$-6 + 2\sqrt{5} + 8$

$\boxed{2 + 2\sqrt{5}}$

Continue question 3 on page 15.

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(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$+ \rightarrow -$

There are critical numbers at $x = -1, 1.5$ and 5.

$$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$$

$$g'(x) = f(x) = 0 \text{ at } x = -1, 1.5, 5$$

$+ \quad - \quad + \quad -$

$-1 \quad 1.5 \quad 5$

$$g(-1) = \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$$

$$g(5) = \int_{-2}^{5} f(t)\,dt = 11 - \frac{9\pi}{4}$$

Since $g(5) > g(-1)$, the absolute maximum value of g is $11 - \frac{9\pi}{4}$ at $t = 5$. This value is an end point and it ~~may~~ change from positive to negative, alluding to a maximum (absolute) value. It is at a critical value

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

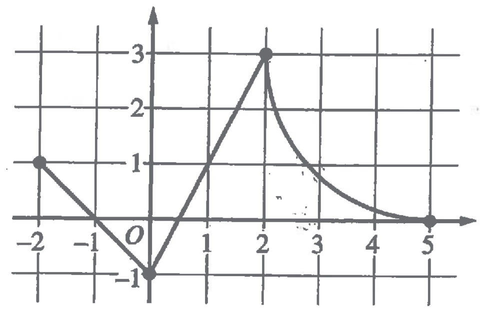
$$f'(1) = \frac{3+1}{2-0} = 2$$

plug in 1

$$\frac{10^1 - 3f'(1)}{f(1) - \tan^{-1}(1)} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}} = \frac{4}{\frac{\pi}{4}} = \boxed{\frac{1}{\pi}}$$

NO CALCULATOR ALLOWED

3C 1of2



Graph of f

$x^2+y^2=r^2$

$x^2+y^2=3^2$

$3^2+(3-\sqrt{5})^2=9$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$\int_{-2}^{5} f(x)\,dx = \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) - \frac{1}{2}(1)(\frac{1}{2}) + \frac{1}{2}(\frac{3}{2})(3) +$

(b) Evaluate $\int_{3}^{5} (2f'(x)+4)\,dx$.

$\int_3^5 2f'(x)+4\ dx$

$[f(x)]^2 + 4x \Big|_3^5$

$[f(5)]^2 + 4(5) - ([f(3)]^2 + 4(3))$

$0 + 20 - (3-\sqrt{5})^2 - 12$

$8 - 14 + 6\sqrt{5} \rightarrow \boxed{-6+6\sqrt{5}}$

$(3-\sqrt{5})(3-\sqrt{5})$

$9 - 3\sqrt{5} - 3\sqrt{5} + 5$

$14 - 6\sqrt{5}$

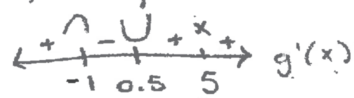
Continue question 3 on page 15.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$$

$$g'(x) = f(x) = 0$$

$$x = -1,\ 0.5,\ 5$$



$$g(-1) = \int_{-2}^{-1} f(t)\,dt$$

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$. $\rightarrow \dfrac{10^1 - 3f'(1)}{f(1) - \arctan(1)} = \dfrac{10 - 3(2)}{1 - \frac{\pi}{4}} = \dfrac{4}{\frac{4-\pi}{4}}$

$$\frac{4}{1} \cdot \frac{4}{4-\pi} = \boxed{\frac{16}{4-\pi}}$$

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem it is given that the function f is continuous on the interval $[-6, 5]$. The portion of the graph of f corresponding to $-2 \le x \le 5$ consists of two line segments and a quarter of a circle, as shown in an accompanying figure. It is noted that the point $(3, 3 - \sqrt{5})$ is on the quarter circle.

In part (a) students were asked to evaluate $\int_{-6}^{-2} f(x)\,dx$, given that $\int_{-6}^{5} f(x)\,dx = 7$. A response should demonstrate the integral property that $\int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx = \int_{-6}^{5} f(x)\,dx$ and use the interpretation of the integral in terms of the area between the graph of f and the x-axis to evaluate $\int_{-2}^{5} f(x)\,dx$ from the given graph.

In part (b) students were asked to evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$. A response should demonstrate the sum and constant multiple properties of definite integrals, together with an application of the Fundamental Theorem of Calculus that gives $\int_{3}^{5} f'(x)\,dx = f(5) - f(3)$.

In part (c) students were asked to find the absolute maximum value for the function g given by $g(x) = \int_{-2}^{x} f(t)\,dt$ on the interval $-2 \le x \le 5$. A response should demonstrate calculus techniques for optimizing a function, starting by applying the Fundamental Theorem of Calculus to obtain $g'(x) = f(x)$, and then using the supplied portion of the graph of f to find critical points for g and to evaluate g at these critical points and the endpoints of the interval.

In part (d) students were asked to evaluate $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$. A response should demonstrate the application of properties of limits, using the supplied portion of the graph of f to evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} f'(x)$.

For part (a) see LO FUN-6.A/EK FUN-6.A.2, LO FUN-6.A/EK FUN-6.A.1. For part (b) see LO FUN-6.B/EK FUN-6.B.2. For part (c) see LO FUN-5.A/EK FUN-5.A.2, LO FUN-4.A/EK FUN-4.A.3. For part (d) see LO LIM-1.D/EK LIM-1.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ given for $\int_{-2}^{5} f(x)\,dx$ in line 2. The third point was earned with the answer

Question 3 (continued)

$7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ in line 2. Numerical simplification of the expression is not required. In part (b) the response earned the first point with $2 \cdot [[f(5) + 2(5)] - [f(3) + 2(3)]]$ in line 4. Note that $2 \cdot [f(x) + 2x]_3^5$ in line 3 is not sufficient to earn the first point. The second point was earned with the answer of $2 \cdot [(0 + 10) - ((3 - \sqrt{5}) + 6)]$ in the last line. Numerical simplification is not required. In part (c) the first point was earned with $g'(x) = f(x)$ in line 1 on the left. The second point was earned in line 3 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values are required for the third point. The response earned the third point by declaring the absolute maximum value of $11 - \frac{9\pi}{4}$ in line 9 and justifying this answer with the labeled values of g for both critical points and both endpoints. The statement at the top right that references the EVT (Extreme Value Theorem) is not required for the point. In part (d) the point was earned with the answer $\frac{4}{1 - \frac{\pi}{4}}$ in the last line. Numerical simplification is not required, and the work presented in lines 1 and 2 is not required to earn the point.

Sample: 3B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ given for $\int_{-2}^{5} f(x)\,dx$ in line 3. The third point would have been earned with the answer $7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ in line 3. The numerical simplification to $-2 + \frac{9\pi}{4}$ in line 3 is incorrect, so the third point was not earned. In part (b) the response earned the first point with $2(f(5) - f(3)) + 20 - 12$ in line 3 on the left. Note that $2[f(x)]_3^5 + [4x]_3^5$ in line 2 on the left is not sufficient to earn the first point. The second point would have been earned for $2(0 - 3 + \sqrt{5}) + 8$ in line 4 on the left. Numerical simplification is not required. The boxed answer $2 + 2\sqrt{5}$ is correct, so the second point was earned. In part (c) the first point was earned with $g'(x) = f(x)$ in line 2 on the left. The inclusion of "$= 0$" is not required to earn the point. The second point was earned in line 2 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. The absolute maximum value of $11 - \frac{9\pi}{4}$ is identified in line 4 on the left and declared as the absolute maximum in lines 5 and 6. The third point was not earned because of an insufficient justification. An incorrect critical point is declared at $x = 1.5,$ and the sign chart presented without explanation is not a sufficient justification for elimination of the second critical point. In part (d) the point would have been earned with the answer $\frac{10 - 3(2)}{1 - \frac{\pi}{4}}.$ Numerical simplification is not required, though the result of $\frac{1}{\pi}$ is incorrect. The point was not earned.

Question 3 (continued)

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the property of definite integrals that is required is not stated, so the first point was not earned. Although the response begins to calculate $\int_{-2}^{5} f(x)\,dx,$ the work is incomplete, and the second point was not earned. The response is not eligible for the third point. In part (b) the antiderivative of $2f'(x)$ is reported incorrectly as $[f(x)]^2$ in line 2 on the left. The Fundamental Theorem of Calculus is not applied correctly, so the first point was not earned. Because the use of the Fundamental Theorem of Calculus is incorrect, the response is not eligible for the second point. In part (c) the first point was earned in line 2 with $g'(x) = f(x).$ The inclusion of "$= 0$" in line 2 is not required to earn the point. The second point was earned in line 3 with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. An absolute maximum value of g is not given, so the third point was not earned. Note that the sign chart without explanation is not a sufficient justification. In part (d) the point would have been earned with the answer $\dfrac{10 - 3(2)}{1 - \frac{\pi}{4}}$ in line 1. Numerical simplification is not required, though the boxed result of $\dfrac{16}{4 - \pi}$ is correct and earned the point.