2019

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356,$ and $E(t) - L(t) < 0$ for $6.20356 < t \le 8.$ Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3: \begin{cases} 1: \text{sets } E(t) - L(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0,$ the rate of change in the number of fish is decreasing at time $t = 5.$

$2: \begin{cases} 1: \text{considers } E'(5) \text{ and } L'(5) \\ 1: \text{answer with explanation} \end{cases}$

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$\int_0^5 E(t)\,dt \approx 153$ fish

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$\frac{1}{5}\int_0^5 L(t)\,dt \approx \frac{30.295}{5} \approx 6.059$ fish per hour leave the lake

Continue question 1 on page 5.

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

$E(t)-L(t)=0$

$t \approx 6.204$

$E(t)-L(t)$ is the rate at which the number of fish is changing

At time $t=6.204$, the greatest number of fish in the 8 hour period are in the lake. This is because $E(t)-L(t)$ is positive from $t=0$ to $t=6.204$ indicating that the number of fish in the lake is increasing over $(0, 6.204)$, but $E(t)-L(t)$ is negative from $t=6.204$ to $t=8$, which means the number of fish are decreasing in this time period, so the number of fish in the lake is greatest at $t=6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

$\frac{d}{dt}\left(16+15\sin\left(\frac{\pi t}{6}\right)-2^{0.1t^2}\right)\Big|_{t=5} \approx -10.723 \text{ fish/hour}^2$

Since the derivative of $E(t)-L(t)$ at $t=5$ is negative, the rate of change in the number of fish in the lake is decreasing

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$$\int_0^5 E(t)\,dt = 153.458$$

153 fish enter over 5-hour period

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$$\frac{1}{5}\int_0^5 L(t)\,dt = 6.059$$

Continue question 1 on page 5.

1B
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(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

max

$E(t) - L(t) = 0$ and changes signs (+) to (−)

$t = 6.204$

There is the greatest number of fish in the lake at time $t = 6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

$E(5) - L(5) = 17.843$

The rate of change in the number of fish in the lake is increasing since $E(5) - L(5) > 0$ and represents the rate of change.

GO ON TO THE NEXT PAGE.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

$$\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \quad \cancel{\int_0^5 4 + 2^{0.1t^2}}$$

153.458 or $\boxed{153}$

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

$$\int_0^5 \left(4 + 2^{0.1t^2}\right)dt \rightarrow \boxed{30.295}$$

Continue question 1 on page 5.

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(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

~~8~~ 5 because $\int_0^8 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt\ \int_0^8 \left(4 + 2^{0.1t^2}\right) dt$ 80.92

and $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt\ \int_0^8 \left(4 + 2^{0.1t^2}\right) dt = 123.163$

$123.163 > 80.92$

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

Increasing because at 4 A.M. 100

$\left(\int_0^4 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt\ \int_0^4 \left(4 + 2^{0.1t^2}\right) dt = 100.837 \right)$

fish are in the lake but at

5 A.M. → 123.163 fish are in the

lake $\left(\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt - \int_0^5 \left(4 + 2^{0.1t^2}\right) dt = 123.163 \right)$

$123.163 > 100.837$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem, fish enter and leave a lake at rates modeled by functions E and L given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$ and $L(t) = 4 + 2^{0.1t^2}$, respectively. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight $(t = 0)$.

In part (a) students were asked to find the number of fish entering the lake between midnight $(t = 0)$ and 5 A.M. $(t = 5)$ and to provide the answer rounded to the nearest whole number. A response should demonstrate an understanding that a definite integral of the rate at which fish enter the lake over the time interval $0 \le t \le 5$ gives the number of fish that enter the lake during that time period. The numerical value of the integral $\int_0^5 E(t)\,dt$ should be obtained using a graphing calculator.

In part (b) students were asked for the average number of fish that leave the lake per hour over the 5-hour period $0 \le t \le 5$. A response should demonstrate that "number of fish per hour" is a rate, so the question is asking for the average value of $L(t)$ across the interval $0 \le t \le 5$, found by dividing the definite integral of L across the interval by the width of the interval. The numerical value of the expression $\frac{1}{5}\int_0^5 L(t)\,dt$ should be obtained using a graphing calculator.

In part (c) students were asked to find, with justification, the time t in the interval $0 \le t \le 8$ when the population of fish in the lake is greatest. The key understanding here is that the rate of change of the number of fish in the lake, in number of fish per hour, is given by the difference $E(t) - L(t)$. Analysis of this difference using a graphing calculator shows that, for $0 \le t \le 8$, the difference has exactly one sign change, occurring at $t = 6.20356$. Before this time, $E(t) - L(t) > 0$, so the number of fish in the lake is increasing; after this time, $E(t) - L(t) < 0$, so the number of fish in the lake is decreasing. Thus the number of fish in the lake is greatest at $t = 6.204$ (or 6.203). An alternative justification uses the definite integral of $E(t) - L(t)$ over an interval starting at $t = 0$ to find the net change in the number of fish in the lake from time $t = 0$. The candidates for when the fish population is greatest are the endpoints of the time interval $0 \le t \le 8$ and the one time when $E(t) - L(t) = 0$, namely $t = 6.20356$. Numerical evaluation of the appropriate definite integrals on a graphing calculator shows that the number of fish in the lake is greatest at $t = 6.204$ (or 6.203).

In part (d) students were asked whether the rate of change in the number of fish in the lake is increasing or decreasing at time $t = 5$. A response should again demonstrate the understanding that the rate of change of the number of fish in the lake is given by the difference $E(t) - L(t)$, and whether this rate is increasing or decreasing at time $t = 5$ can be determined by the sign of the derivative of the difference at that time. Using a graphing calculator to find that $E'(5) - L'(5) < 0$ leads to the conclusion that the rate of change in the number of fish in the lake is decreasing at time $t = 5$.

Question 1 (continued)

For part (a) see LO CHA-4.E/EK CHA-4.E.1, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1. For part (c) see LO FUN-4.B/EK FUN-4.B.1. For part (d) see LO CHA-3.C/EK CHA-3.C.1, LO CHA-2.D/EK CHA-2.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1 on the left. The sentence "[a]t time $t = 6.204,$ the greatest number of fish in the 8 hour period are in the lake" in lines 3 and 4 would have earned the second point without additional information. The second point was earned with the restatement "so the number of fish in the lake is greatest at $t = 6.204$ hours" in lines 7 and 8. The third point was earned with the statements "because $E(t) - L(t)$ is positive from $t = 0$ to $t = 6.204$" and "$E(t) - L(t)$ is negative from $t = 6.204$ to $t = 8$" in lines 4, 5, and 6. In part (d) the response earned the first point in line 1 with $16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2},$ which is equivalent to $E(t) - L(t),$ and $\frac{d}{dt}\left(16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2}\right)\Big|_{t=5},$ which is equivalent to $E'(5) - L'(5).$ The second point was earned with "decreasing" and the explanation, "[s]ince the derivative of $E(t) - L(t)$ at $t = 5$ is negative" in the concluding sentence.

Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1. The second point was earned with "[t]here is the greatest number of fish in the lake at time $t = 6.204$" in lines 3 and 4. The third point was not earned because the statement "changes signs $(+)$ to $(-)$" and $t = 6.204$ in lines 1 and 2 only provides evidence that there is a relative maximum at $t = 6.204$ rather than an absolute maximum on the interval $0 \le t \le 8.$ In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "increasing" is incorrect.

Question 1 (continued)

Sample: 1C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right) dt,$ and the second point was earned with the answer $153.$ The crossed-out work is not scored. In part (b) the first point was earned with the definite integral $\int_0^5 \left(4 + 2^{0.1t^2}\right) dt.$ The second point was not earned because the integral is not multiplied by $\frac{1}{5};$ the answer is incorrect. In part (c) no points were earned because there is no equating of $E(t) - L(t)$ to $0,$ and there is no declaration of an absolute maximum value nor a justification. In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "Increasing" is incorrect.