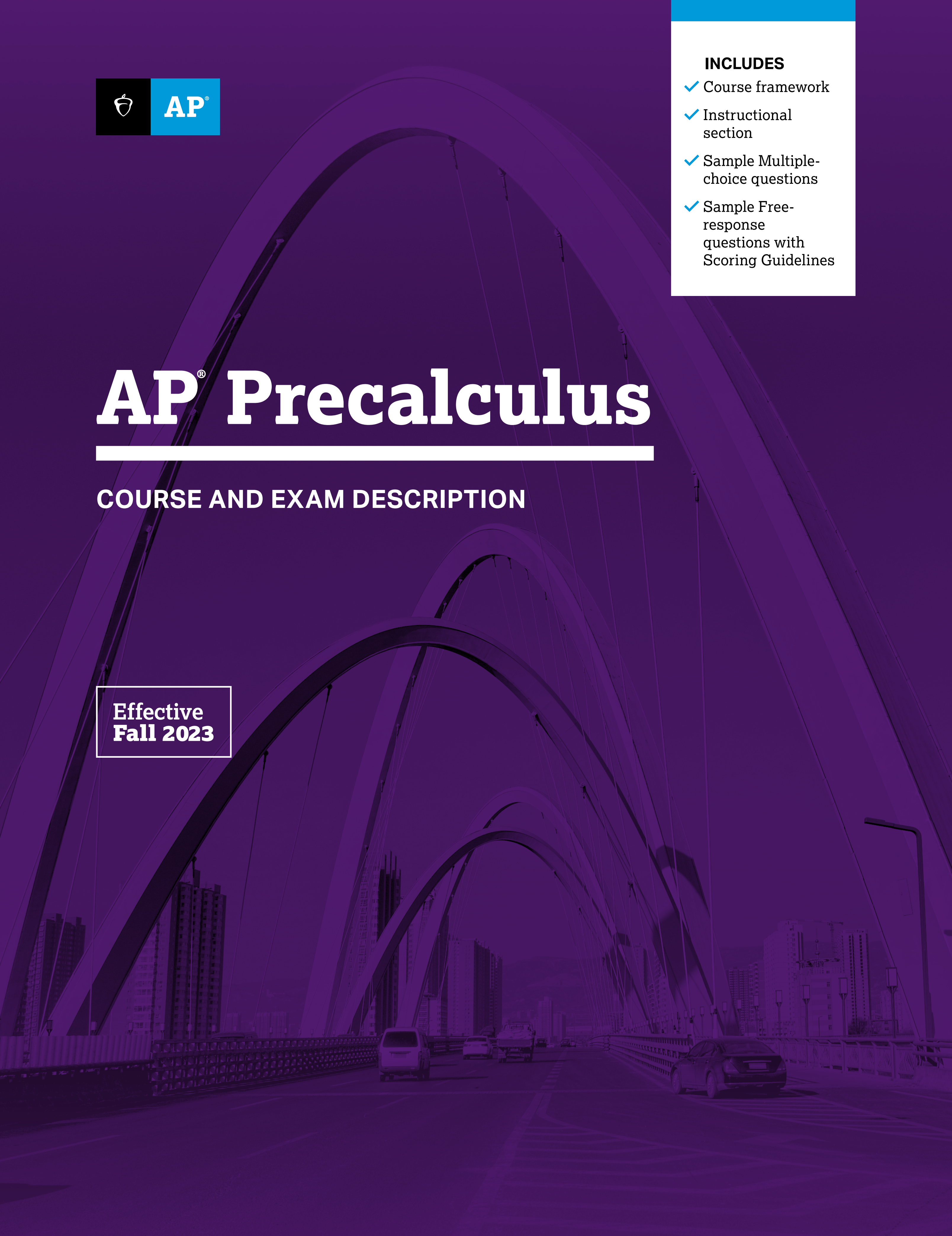

INCLUDES

- ✓ Course framework
- ✓ Instructional section
- ✓ Sample Multiple-choice questions
- ✓ Sample Free-response questions with Scoring Guidelines

AP® Precalculus

COURSE AND EXAM DESCRIPTION

Effective
Fall 2023

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AP® Precalculus

COURSE AND EXAM DESCRIPTION

Effective
Fall 2023

AP COURSE AND EXAM DESCRIPTIONS ARE UPDATED PERIODICALLY
Please visit AP Central (apcentral.collegeboard.org) to determine whether a more recent course and exam description is available.

What AP® Stands For

Thousands of Advanced Placement teachers have contributed to the principles articulated here. These principles are not new; they are, rather, a reminder of how AP already works in classrooms nationwide. The following principles are designed to ensure that teachers' expertise is respected, required course content is understood, and that students are academically challenged and free to make up their own minds.

1. AP stands for clarity and transparency. Teachers and students deserve clear expectations. The Advanced Placement Program makes public its course frameworks and sample assessments. Confusion about what is permitted in the classroom disrupts teachers and students as they navigate demanding work.
2. AP is an unflinching encounter with evidence. AP courses enable students to develop as independent thinkers and to draw their own conclusions. Evidence and the scientific method are the starting place for conversations in AP courses.
3. AP opposes censorship. AP is animated by a deep respect for the intellectual freedom of teachers and students alike. If a school bans required topics from their AP courses, the AP Program removes the AP designation from that course and its inclusion in the AP Course Ledger provided to colleges and universities. For example, the concepts of evolution are at the heart of college biology, and a course that neglects such concepts does not pass muster as AP Biology.
4. AP opposes indoctrination. AP students are expected to analyze different perspectives from their own, and no points on an AP Exam are awarded for agreement with a viewpoint. AP students are not required to feel certain ways about themselves or the course content. AP courses instead develop students' abilities to assess the credibility of sources, draw conclusions, and make up their own minds.

   As the AP English Literature course description states: "AP students are not expected or asked to subscribe to any one specific set of cultural or political values, but are expected to have the maturity to analyze perspectives different from their own and to question the meaning, purpose, or effect of such content within the literary work as a whole."
5. AP courses foster an open-minded approach to the histories and cultures of different peoples. The study of different nationalities, cultures, religions, races, and ethnicities is essential within a variety of academic disciplines. AP courses ground such studies in primary sources so that students can evaluate experiences and evidence for themselves.
6. Every AP student who engages with evidence is listened to and respected. Students are encouraged to evaluate arguments but not one another. AP classrooms respect diversity in backgrounds, experiences, and viewpoints. The perspectives and contributions of the full range of AP students are sought and considered. Respectful debate of ideas is cultivated and protected; personal attacks have no place in AP.
7. AP is a choice for parents and students. Parents and students freely choose to enroll in AP courses. Course descriptions are available online for parents and students to inform their choice. Parents do not define which college-level topics are suitable within AP courses; AP course and exam materials are crafted by committees of professors and other expert educators in each field. AP courses and exams are then further validated by the American Council on Education and studies that confirm the use of AP scores for college credits by thousands of colleges and universities nationwide.

The AP Program encourages educators to review these principles with parents and students so they know what to expect in an AP course. Advanced Placement is always a choice, and it should be an informed one. AP teachers should be given the confidence and clarity that once parents have enrolled their child in an AP course, they have agreed to a classroom experience that embodies these principles.

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SCORING GUIDELINES

APPENDIX

Acknowledgments

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About AP

The Advanced Placement® Program (AP®) enables willing and academically prepared students to pursue college-level studies—with the opportunity to earn college credit, advanced placement, or both—while still in high school. Through AP courses in 39 subjects, each culminating in a challenging exam, students learn to think critically, construct solid arguments, and see many sides of an issue—skills that prepare them for college and beyond. Taking AP courses demonstrates to college admission officers that students have sought the most challenging curriculum available to them, and research indicates that students who score a 3 or higher on an AP Exam typically experience greater academic success in college and are more likely to earn a college degree than non-AP students. Each AP teacher's syllabus is evaluated and approved by faculty from some of the nation's leading colleges and universities, and AP Exams are developed and scored by college faculty and experienced AP teachers. Most four-year colleges and universities in the United States grant credit, advanced placement, or both on the basis of successful AP Exam scores—more than 3,300 institutions worldwide annually receive AP scores.

AP Course Development

In an ongoing effort to maintain alignment with best practices in college-level learning, AP courses and exams emphasize challenging, research-based curricula aligned with higher education expectations.

Individual teachers are responsible for designing their own curriculum for AP courses, selecting appropriate college-level readings, assignments, and resources. This course and exam description presents the content and skills that are the focus of the corresponding college course and that appear on the AP Exam. It also organizes the content and skills into a series of units that represent a sequence found in widely adopted college textbooks and that many AP teachers have told us they follow in order to focus their instruction. The intention of this publication is to respect teachers' time and expertise by providing a roadmap that they can modify and adapt to their local priorities and preferences. Moreover, by organizing the AP course content and skills into units, the AP Program is able to provide teachers and students with free formative assessments—Progress Checks—that teachers can assign throughout the year to measure student progress as they acquire content knowledge and develop skills.

Enrolling Students: Access, Opportunity, and Readiness

The AP Program welcomes all students willing to challenge themselves with college-level coursework and career preparation. We strongly encourage educators to invite students into AP classes, including students from ethnic, racial, socioeconomic, geographic, or other groups not broadly participating in a school's AP program. We believe that readiness for AP is attainable, and that educators can expand readiness by opening access to Pre-AP course work. We commit to support educators and communities in their efforts to make AP courses widely available, advancing students in their plans for college and careers.

Offering AP Courses: The AP Course Audit

The AP Program unequivocally supports the principle that each school implements its own curriculum that will enable students to develop the content understandings and skills described in the course framework.

While the unit sequence represented in this publication is optional, the AP Program does have a short list of curricular and resource requirements that must be fulfilled before a school can label a course "Advanced Placement" or "AP." Schools wishing to offer AP courses must participate in the AP Course Audit, a process through which AP teachers' course materials are reviewed by college faculty. The AP Course Audit was created to provide teachers and administrators with clear guidelines on curricular and resource requirements for AP courses and to help colleges and universities validate courses marked "AP" on students' transcripts. This process ensures that AP teachers' courses meet or exceed the curricular and resource expectations that college and secondary school faculty have established for college-level courses.

The AP Course Audit form is submitted by the AP teacher and the school principal (or designated administrator) to confirm awareness and understanding of the curricular and resource requirements. A syllabus or course outline, detailing how course requirements are met, is submitted by the AP teacher for review by college faculty.

Please visit the **AP Course Audit** website for more information to support the preparation and submission of materials for the AP Course Audit.

How the AP Program Is Developed

The scope of content for an AP course and exam is derived from an analysis of hundreds of syllabi and course offerings of colleges and universities. Using this research and data, a committee of college faculty and expert AP teachers work within the scope of the corresponding college course to articulate what students should know and be able to do upon the completion of the AP course. The resulting course framework is the heart of this course and exam description and serves as a blueprint of the content and skills that can appear on an AP Exam.

The AP Test Development Committees are responsible for developing each AP Exam, ensuring the exam questions are aligned to the course framework. The AP Exam development process is a multiyear endeavor; all AP Exams undergo extensive review, revision, piloting, and analysis to ensure that questions are accurate, fair, and valid, and that there is an appropriate spread of difficulty across the questions.

Committee members are selected to represent a variety of perspectives and institutions (public and private, small and large schools and colleges), and a range of gender, racial/ethnic, and regional groups. A list of each subject's current AP Test Development Committee members is available on **AP Central®**.

Throughout AP course and exam development, College Board gathers feedback from various stakeholders in both secondary schools and higher education institutions. This feedback is carefully considered to ensure that AP courses and exams are able to provide students with a college-level learning experience and the opportunity to demonstrate their qualifications for advanced placement or college credit.

How AP Exams Are Scored

The exam scoring process, like the course and exam development process, relies on the expertise of both AP teachers and college faculty. While multiple-choice questions are scored by machine, the free-response questions and through-course performance assessments, as applicable, are scored by thousands of college faculty and expert AP teachers. Most are scored at the annual AP Reading, while a small portion is scored online. All AP Readers are thoroughly trained, and their work is monitored throughout the Reading for fairness and consistency. In each subject, a highly respected college faculty member serves as Chief Faculty Consultant and, with the help of AP Readers in leadership positions, maintains the accuracy of the scoring standards. Scores on the free-response questions and performance assessments are weighted and combined with the results of the computer-scored multiple-choice questions, and this raw score is converted into a composite AP score on a 1–5 scale.

AP Exams are **not** norm-referenced or graded on a curve. Instead, they are criterion-referenced, which means that every student who meets the criteria for an AP score of 2, 3, 4, or 5 will receive that score, no matter how many students that is. The criteria for the number of points students must earn on the AP Exam to receive scores of 3, 4, or 5—the scores that research consistently validates for credit and placement purposes—include:

- The number of points successful college students earn when their professors administer AP Exam questions to them.
- The number of points researchers have found to be predictive that an AP student will succeed when placed into a subsequent higher-level college course.
- Achievement-level descriptions formulated by college faculty who review each AP Exam question.

Using and Interpreting AP Scores

The extensive work done by college faculty and AP teachers in the development of the course and exam and throughout the scoring process ensures that AP Exam scores accurately represent students' achievement in the equivalent college course. Frequent and regular research studies establish the validity of AP scores as follows:

AP Score	Credit Recommendation	College Grade Equivalent
5	Extremely well qualified	A
4	Well qualified	A–, B+, B
3	Qualified	B–, C+, C
2	Possibly qualified	n/a
1	No recommendation	n/a

While colleges and universities are responsible for setting their own credit and placement policies, most private colleges and universities award credit and/or advanced placement for AP scores of 3 or higher. Additionally, most states in the U.S. have adopted statewide credit policies that ensure college credit for scores of 3 or higher at public colleges and universities. To confirm a specific college's AP credit/placement policy, use the search engine available on the **AP Credit Policy Search** page.

BECOMING AN AP READER

Each June, thousands of AP teachers and college faculty members from around the world gather for seven days in multiple locations to evaluate and score the free-response sections of the AP Exams. Ninety-eight percent of surveyed educators who took part in the AP Reading say it was a positive experience.

There are many reasons to consider becoming an AP Reader, including opportunities to:

- **Bring positive changes to the classroom:** Surveys show that the vast majority of returning AP Readers—both high school and college educators—make improvements to the way they teach or score because of their experience at the AP Reading.
- **Gain in-depth understanding of AP Exam and AP scoring standards:** AP Readers gain exposure to the quality and depth of the responses from the entire pool of AP Exam takers, and thus are better able to assess their students' work in the classroom.
- **Receive compensation:** AP Readers are compensated for their work during the Reading. Expenses, lodging, and meals are covered for Readers who travel.
- **Score from home:** AP Readers have online distributed scoring opportunities for certain subjects. Check the **AP Reader** site for details.
- **Earn Continuing Education Units (CEUs):** AP Readers earn professional development hours and CEUs that can be applied to PD requirements by states, districts, and schools.

How to Apply

Visit the **Become an AP Reader** site for eligibility requirements and to start the application process.

AP Resources and Supports

By completing a simple class selection process at the start of the school year, teachers and students receive access to a robust set of classroom resources.

AP Classroom

AP Classroom is a dedicated online platform designed to support teachers and students throughout their AP experience. The platform provides a variety of powerful resources and tools to provide yearlong support to teachers and students, offering opportunities to give and get meaningful feedback on student progress.

UNIT GUIDES

Appearing in this publication and on AP Classroom, these planning guides outline all required course content and skills, organized into commonly taught units. Each Unit Guide suggests a sequence and pacing of content, scaffolds skill instruction across units, organizes content into topics, and provides tips on taking the AP Exam.

PROGRESS CHECKS

Formative AP questions for every unit provide feedback to students on the areas where they need to focus. Available online, Progress Checks measure knowledge and skills through multiple-choice questions with rationales to explain correct and incorrect answers, and free-response questions with scoring information. Because the Progress Checks are formative, the results of these assessments cannot be used to evaluate teacher effectiveness or assign letter grades to students, and any such misuses are grounds for losing school authorization to offer AP courses.*

MY REPORTS

My reports provides teachers with a one-stop shop for student results on all assignment types, including Progress Checks. Teachers can view class trends and see where students struggle with content and skills that will be assessed on the AP Exam. Students can view their own progress over time to improve their performance before the AP Exam.

QUESTION BANK

The Question Bank is a searchable library of all AP questions that teachers use to build custom practice for their students. Teachers can create and assign assessments with formative topic questions or questions from practice or released AP Exams.

Class Section Setup and Enrollment

- Teachers and students sign in to or create their College Board accounts.
- Teachers confirm that they have added the course they teach to their AP Course Audit account and have had it approved by their school's administrator.
- Teachers or AP coordinators, depending on who the school has decided is responsible, set up class sections so students can access AP resources and have exams ordered on their behalf.
- Students join class sections with a join code provided by their teacher or AP coordinator.
- Students will be asked for additional information upon joining their first class section.

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* To report misuses, please call, 877-274-6474 (International: 212-632-1781).

Instructional Model

Integrating AP resources throughout the course can help students develop skills and conceptual understandings. The instructional model outlined below shows possible ways to incorporate AP resources into the classroom.

Plan

Teachers may consider the following approaches as they plan their instruction before teaching each unit.

- Review the overview at the start of each **Unit Guide** to identify essential questions, conceptual understandings, and skills for each unit.
- Use the **Unit at a Glance** table to identify related topics that build toward a common understanding, and then plan appropriate pacing for students.
- Identify useful strategies in the **Instructional Approaches** section to help teach the concepts and skills.

Teach

When teaching, supporting resources could be used to build students' conceptual understanding and their mastery of skills.

- Use the topic pages in the **Unit Guides** to identify the required content.
- Integrate the content with a skill, considering any appropriate scaffolding.
- Employ any of the instructional strategies previously identified.
- Use the available resources, including **AP Daily**, on the topic pages to bring a variety of assets into the classroom.

Assess

Teachers can measure student understanding of the content and skills covered in the unit and provide actionable feedback to students.

- As you teach each topic, use **AP Classroom** to assign student **Topic Questions** as a way to continuously check student understanding and provide just in time feedback
- At the end of each unit, use **AP Classroom** to assign students **Progress Checks**, as homework or an in-class task.
- Provide question-level feedback to students through answer rationales; provide unit- and skill-level formative feedback using **My Reports**.
- Create additional practice opportunities using the **Question Bank** and assign them through **AP Classroom**.

About the AP Precalculus Course

AP Precalculus centers on functions modeling dynamic phenomena. This research-based exploration of functions is designed to better prepare students for college-level calculus and provide grounding for other mathematics and science courses. In this course, students study a broad spectrum of function types that are foundational for careers in mathematics, physics, biology, health science, business, social science, and data science. Furthermore, as AP Precalculus may be the last mathematics course of a student's secondary education, the course is structured to provide a coherent capstone experience rather than exclusively focusing on preparation for future courses.

Throughout this course, students develop and hone symbolic manipulation skills, including solving equations and manipulating expressions, for the many function types throughout the course. Students also learn that functions and their compositions, inverses, and transformations are understood through graphical, numerical, analytical, and verbal representations, which reveal different attributes of the functions and are useful for solving problems in mathematical and applied contexts. In turn, the skills learned in this course are widely applicable to situations that involve quantitative reasoning.

AP Precalculus fosters the development of a deep conceptual understanding of functions. Students learn that a function is a mathematical relation that maps a set of input values—the domain—to a set of output values—the range—such that each input value is uniquely mapped to an output value. Students understand functions and their graphs as embodying dynamic covariation of quantities, a key idea in preparing for calculus. With each function type, students develop and validate function models based on the characteristics of a bivariate data set, characteristics of covarying quantities and their relative rates of change, or a set of characteristics such as zeros, asymptotes, and extrema. These models are used to interpolate, extrapolate, and interpret information with different degrees of accuracy for a given context or data set. Additionally, students also learn that every model is subject to assumptions and limitations related to the context. As a result of examining functions from many perspectives, students develop a conceptual understanding not only of specific function types but also of functions in general. This type of understanding helps students to engage with both familiar and novel contexts.

College Course Equivalent

AP Precalculus is designed to be the equivalent of a first semester college precalculus course. AP Precalculus provides students with an understanding of the concepts of college algebra, trigonometry, and additional topics that prepare students for further college-level mathematics courses. This course explores a variety of function types and their applications—polynomial, rational, exponential, logarithmic, trigonometric, polar, parametric, vector-valued, implicitly defined, and linear transformation functions using matrices. Throughout the course, the mathematical practices of procedural and symbolic fluency, multiple representations, and communication and reasoning are developed. Students experience the concepts and skills related to each function type through the lenses of modeling and covariation and engage each function type through their graphical, numerical, analytical, and verbal representations.

Prerequisites

Before studying precalculus, all students should develop proficiency in topics typically found in the Algebra 1-Geometry-Algebra 2 (AGA) content sequence. Students should have developed the following:

- Proficiency with the skills and concepts related to linear and quadratic functions, including algebraic manipulation, solving equations, and solving inequalities
- Proficiency in manipulating algebraic expressions related to polynomial functions, including polynomial addition and multiplication, factoring quadratic trinomials, and using the quadratic formula
- Proficiency in solving right triangle problems involving trigonometry
- Proficiency in solving systems of equations in two and three variables
- Familiarity with piecewise-defined functions
- Familiarity with exponential functions and rules for exponents
- Familiarity with radicals (e.g., square roots, cube roots)
- Familiarity with complex numbers
- Familiarity with communicating and reasoning among graphical, numerical, analytical, and verbal representations of functions

Technology Needs

Technology should be used throughout the course as a tool to explore concepts. In AP Precalculus, students should specifically practice using technology to do the following:

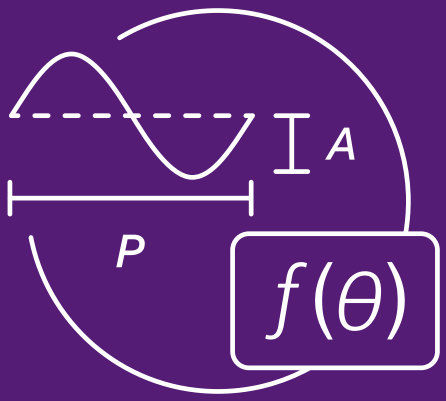
- Perform calculations (e.g., exponents, roots, trigonometric values, logarithms)
- Graph functions and analyze graphs
- Generate a table of values for a function
- Find real zeros of functions
- Find points of intersection of graphs of functions
- Find minima/maxima of functions
- Find numerical solutions to equations in one variable
- Find regression equations to model data (linear, quadratic, cubic, quartic, exponential, logarithmic, and sinusoidal) and plot the corresponding residuals
- Perform matrix operations (e.g., multiplication, finding inverses)

It is important to note that technology should not replace the development of symbolic manipulation skills. When algebraic expressions and equations are accessible with precalculus-level algebraic manipulation, students are expected to find zeros, solve equations, and calculate values without the help of technology. Most of the AP Exam will need to be completed without the use of technology. However, selected multiple-choice and free-response questions will require students to use a graphing calculator to complete the tasks delineated above.

Accessible technology that has the capabilities expected for AP Precalculus is available for students who are blind or visually impaired. This technology should be used during the course, and an accommodation request to use this technology on the AP Exam must be made through the College Board's **Services for Students with Disabilities (SSD)**.

AP PRECALCULUS

Course Framework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Introduction

The course framework for AP Precalculus is partitioned into four units. Units 1, 2, and 3 are required and assessed on the end-of-course AP Exam and provide descriptions of what students should know and be able to do to qualify for college credit or placement. Unit 4 describes additional topics that teachers may include based on state or local requirements. The Unit 4 topics extend and deepen the function concepts developed in units one through three. The Unit 4 topics are additional and excluded from the AP Exam.

The framework is organized by commonly taught units of instruction, informed by extensive research. Teachers may adjust the suggested sequencing of units or topics, although they will want to carefully consider how to account for such changes as they access course resources for planning, teaching, and assessing.

Course Framework Components

Overview

The course framework provides clear and detailed descriptions for the mathematical practices and course content included in the course. Units 1, 2, and 3, and the mathematical practices, specify what students should know, and be able to do for college credit or placement as assessed on the AP Exam. Unit 4 topics are additional topics that teachers may include based on state or local requirements and are excluded from the AP Exam.

The course framework includes two essential components:

① MATHEMATICAL PRACTICES

The mathematical practices are central to the study and practice of precalculus. Students should develop and apply the described skills on a regular basis over the span of the course.

② COURSE CONTENT

The course content is organized into units of study that provide a suggested sequence for the course. Units 1, 2, and 3 topics comprise the content and conceptual understandings that colleges and universities typically expect students to be proficient in in order to qualify for college credit and/or placement, and are therefore included on the AP Exam. Unit 4 consists of topics that teachers may include based on state or local requirements.

COURSE FRAMEWORK CONVENTIONS:

Common language usage (e.g., “area of a triangle”) replaces precise mathematical phrasing (e.g., “area of the interior of a triangle”) in the following instances:

- When the framework refers to modeling a data set, it is referring to a bivariate data set.
- When the framework refers to modeling a context or phenomenon, it is referring to two aspects of the context or phenomenon.
- When the framework refers to the sine, cosine, and so on of an angle, it is referring to the sine, cosine, and so on of the measure of the angle.

1

AP PRECALCULUS

Mathematical Practices

The eight distinct skills are associated with three mathematical practices. Students should build and master these skills throughout the course. While many different skills can be applied to any one content topic, the framework supplies skill focus recommendations for each topic to help assure skill distribution and repetition throughout the course.

More detailed information about teaching the mathematical practices can be found in the **Instructional Approaches** section of this publication.

AP PRECALCULUS

Mathematical Practices

Practice 1

Procedural and Symbolic Fluency 1

Algebraically manipulate functions, equations, and expressions.

SKILLS

1.A **Solve equations and inequalities** represented analytically, with and without technology.

1.B **Express** functions, equations, or expressions in analytically **equivalent forms** that are useful in a given mathematical or applied context.

1.C **Construct new functions**, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

Practice 2

Multiple Representations 2

Translate mathematical information between representations.

2.A **Identify information from** graphical, numerical, analytical, and verbal **representations** to answer a question or construct a model, with and without technology.

2.B **Construct equivalent** graphical, numerical, analytical, and verbal **representations** of functions that are useful in a given mathematical or applied context, with and without technology.

Practice 3

Communication and Reasoning 3

Communicate with precise language, and provide rationales for conclusions.

3.A **Describe** the **characteristics** of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.B **Apply** numerical **results** in a given mathematical or applied context.

3.C **Support conclusions** or choices with a logical rationale or appropriate data.

2

AP PRECALCULUS

Course Content

This course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students must know, be able to do, and understand.

The framework also encourages instruction that prepares students for advanced coursework in mathematics or other fields engaged in modeling change (e.g., pure sciences, engineering, or economics) and for creating useful, reasonable solutions to problems encountered in an ever-changing world.

UNITS

The AP Precalculus content is subdivided into four units. Units 1, 2, and 3 topics are included on the AP Exam. Unit 4 topics are at the discretion of the school and teacher, based on state and local requirements. Pacing recommendations at the unit level and on the Course at a Glance provide suggestions for how teachers can teach the required course content. The suggested class periods are based on a school schedule in which the class meets five days a week for 45 minutes each day, for a full school year.

Many topics within Units 1, 2, and 3 list a range of recommended days. When a range is listed, the lower value is the recommendation based on a teacher including all of the topics in Units 1, 2, 3, and 4. The higher value is the recommendation based on a teacher including topics only in Units 1, 2, and 3. While these recommendations have been made to aid planning, teachers are free to adjust the pacing based on the needs of their students, alternate schedules (e.g., block scheduling), their school's academic calendar, or the extent to which Unit 4 topics are included.

TOPICS

Each unit is broken down into teachable segments called topics. The topic pages (starting on p. 31) contain the required content for each topic. Although most topics can be taught in one to three class periods, teachers should pace the course to suit the needs of their students and school.

Exam Weighting for the Multiple-Choice Section of the AP Exam

Units	Exam Weighting
Unit 1: Polynomial and Rational Functions	**30–40%**
Unit 2: Exponential and Logarithmic Functions	**27–40%**
Unit 3: Trigonometric and Polar Functions	**30–35%**
Unit 4: Functions Involving Parameters, Vectors, and Matrices	**Not assessed on the AP Exam**

Units 1, 2, and 3 topics comprise the content and conceptual understandings in which colleges and universities typically expect students to be proficient, in order to qualify for college credit and/or placement. Therefore, these topics are included on the AP Exam. Unit 4 consists of topics that teachers may include based on state or local requirements.

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Course at a Glance

Plan

The Course at a Glance provides a useful visual organization for the AP Precalculus curricular components, including:

- Sequence of units, along with approximate weighting and suggested pacing. Please note, pacing is based on 45-minute class periods, meeting five days each week for a full academic year.
- Progression of topics within each unit.

Teach

MATHEMATICAL PRACTICES

1 Procedural and Symbolic Fluency

3 Communication and Reasoning

2 Multiple Representations

Required Course Content

Each topic contains required Learning Objectives and Essential Knowledge Statements that form the basis of the assessment on the AP Exam.

Assess

Assign the Progress Checks—either as homework or in class—for each unit. Each Progress Check contains formative multiple-choice and free-response questions. The feedback from the Progress Checks shows students the areas where they need to focus.

UNIT 1 Polynomial and Rational Functions

6–8 weeks | **30–40%** AP Exam Weighting

2 3	**1.1** Change in Tandem
2 3	**1.2** Rates of Change
3	**1.3** Rates of Change in Linear and Quadratic Functions
2 3	**1.4** Polynomial Functions and Rates of Change
1 2	**1.5** Polynomial Functions and Complex Zeros
3	**1.6** Polynomial Functions and End Behavior
1 3	**1.7** Rational Functions and End Behavior
1	**1.8** Rational Functions and Zeros
2	**1.9** Rational Functions and Vertical Asymptotes
3	**1.10** Rational Functions and Holes
1 3	**1.11** Equivalent Representations of Polynomial and Rational Expressions
1 3	**1.12** Transformations of Functions
2 3	**1.13** Function Model Selection and Assumption Articulation
1 3	**1.14** Function Model Construction and Application

Progress Check Unit 1 Part 1: Topics 1.1–1.6

Multiple-choice: 18
Free-response: 2

Progress Check Unit 1 Part 2: Topics 1.7–1.14

Multiple-choice: 24
Free-response: 2

UNIT 2 Exponential and Logarithmic Functions

6–9 weeks | **27–40%** AP Exam Weighting

1 3	**2.1** Change in Arithmetic and Geometric Sequences
1 3	**2.2** Change in Linear and Exponential Functions
3	**2.3** Exponential Functions
1 3	**2.4** Exponential Function Manipulation
1 3	**2.5** Exponential Function Context and Data Modeling
2 3	**2.6** Competing Function Model Validation
1 2	**2.7** Composition of Functions
1 2	**2.8** Inverse Functions
1	**2.9** Logarithmic Expressions
1 2	**2.10** Inverses of Exponential Functions
3	**2.11** Logarithmic Functions
1 3	**2.12** Logarithmic Function Manipulation
1	**2.13** Exponential and Logarithmic Equations and Inequalities
1 3	**2.14** Logarithmic Function Context and Data Modeling
2 3	**2.15** Semi-log Plots

Progress Check Unit 2 Part 1: Topics 2.1–2.8

Multiple-choice: 24
Free-response: 2

Progress Check Unit 2 Part 2: Topics 2.9–2.15

Multiple-choice: 24
Free-response: 2

UNIT 3 Trigonometric and Polar Functions

7–10 weeks | **30–35%** AP Exam Weighting

2 3	**3.1**	Periodic Phenomena
2 3	**3.2**	Sine, Cosine, and Tangent
2 3	**3.3**	Sine and Cosine Function Values
2 3	**3.4**	Sine and Cosine Function Graphs
2 3	**3.5**	Sinusoidal Functions
1 2	**3.6**	Sinusoidal Function Transformations
1 3	**3.7**	Sinusoidal Function Context and Data Modeling
2 3	**3.8**	The Tangent Function
1 2	**3.9**	Inverse Trigonometric Functions
1 2 3	**3.10**	Trigonometric Equations and Inequalities
2 3	**3.11**	The Secant, Cosecant, and Cotangent Functions
1 3	**3.12**	Equivalent Representations of Trigonometric Functions
1 2	**3.13**	Trigonometry and Polar Coordinates
2 3	**3.14**	Polar Function Graphs
3	**3.15**	Rates of Change in Polar Functions

Progress Check Unit 3 Part 1: Topics 3.1–3.7

Multiple-choice: 21
Free-response: 2

Progress Check Unit 3 Part 2: Topics 3.8–3.15

Multiple-choice: 24
Free-response: 2

UNIT 4 Functions Involving Parameters, Vectors, and Matrices

7 weeks | **0%** AP Exam Weighting

1 2	**4.1**	Parametric Functions
3	**4.2**	Parametric Functions Modeling Planar Motion
3	**4.3**	Parametric Functions and Rates of Change
1	**4.4**	Parametrically Defined Circles and Lines
2 3	**4.5**	Implicitly Defined Functions
1 2	**4.6**	Conic Sections
1 2	**4.7**	Parametrization of Implicitly Defined Functions
2 3	**4.8**	Vectors
3	**4.9**	Vector-Valued Functions
1 3	**4.10**	Matrices
1 3	**4.11**	The Inverse and Determinant of a Matrix
1	**4.12**	Linear Transformations and Matrices
1 2 3	**4.13**	Matrices as Functions
1 3	**4.14**	Matrices Modeling Contexts

Progress Check Unit 4 Part 1: Topics 4.1–4.7

Multiple-choice: 24
Free-response: 2

Progress Check Unit 4 Part 2: Topics 4.8–4.14

Multiple-choice: 21
Free-response: 2

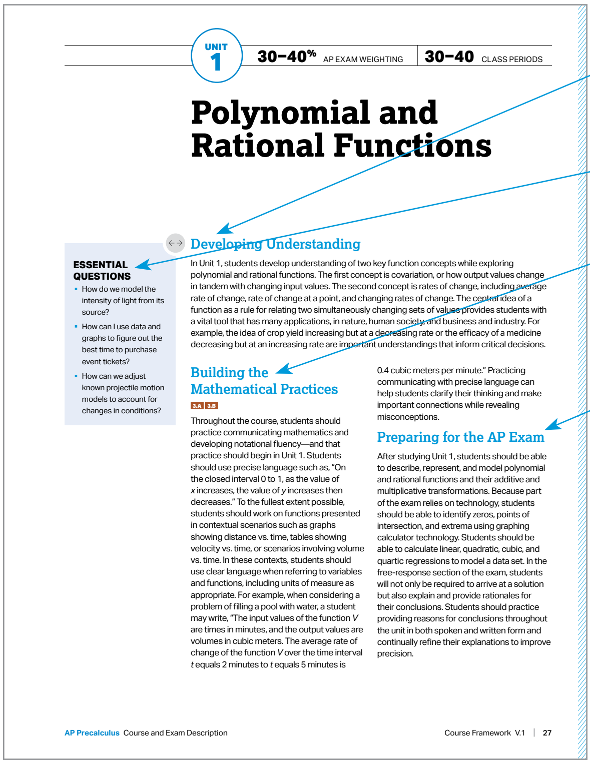
AP PRECALCULUS

Unit Guides

Introduction

Designed with extensive input from the community of AP Precalculus educators, the Unit Guides offer teachers helpful guidance in building students' skills and knowledge. The suggested sequence was identified through a thorough analysis of the syllabi of highly effective AP teachers and the organization of typical college textbooks. This unit structure respects new AP teachers' time by providing one possible sequence they can adopt or modify rather than having to build from scratch. An additional benefit is that these units enable the AP Program to provide interested teachers with formative assessments—the Progress Checks—that they can assign their students at the end of each unit to gauge progress toward success on the AP Exam. However, experienced AP teachers who are satisfied with their current course organization and exam results should feel no pressure to adopt these units, which comprise an optional sequence for this course.

Using the Unit Guides



UNIT 1

30–40% AP EXAM WEIGHTING

30–40 CLASS PERIODS

Polynomial and Rational Functions

ESSENTIAL QUESTIONS

- How do we model the intensity of light from its source?
- How can I use data and graphs to figure out the best time to purchase event tickets?
- How can we adjust known projectile motion models to account for changes in conditions?

Developing Understanding

In Unit 1, students develop understanding of two key function concepts while exploring polynomial and rational functions. The first concept is covariation, or how output values change in tandem with changing input values. The second concept is rates of change, including average rate of change, rate of change at a point, and changing rates of change. The central idea of a function as a rule for relating two simultaneously changing sets of values provides students with a vital tool that has many applications, in nature, human society, and business and industry. For example, the idea of crop yield increasing but at a decreasing rate or the efficacy of a medicine decreasing but at an increasing rate are important understandings that inform critical decisions.

Building the Mathematical Practices

3.A 3.B

Throughout the course, students should practice communicating mathematics and developing notational fluency—and that practice should begin in Unit 1. Students should use precise language such as, "On the closed interval 0 to 1, as the value of *x* increases, the value of *y* increases then decreases." To the fullest extent possible, students should work on functions presented in contextual scenarios such as graphs showing distance vs. time, tables showing velocity vs. time, or scenarios involving volume vs. time. In these contexts, students should use clear language when referring to variables and functions, including units of measure as appropriate. For example, when considering a problem of filling a pool with water, a student may write, "The input values of the function *V* are times in minutes, and the output values are volumes in cubic meters. The average rate of change of the function *V* over the time interval *t* equals 2 minutes to *t* equals 5 minutes is 0.4 cubic meters per minute." Practicing communicating with precise language can help students clarify their thinking and make important connections while revealing misconceptions.

Preparing for the AP Exam

After studying Unit 1, students should be able to describe, represent, and model polynomial and rational functions and their additive and multiplicative transformations. Because part of the exam relies on technology, students should be able to identify zeros, points of intersection, and extrema using graphing calculator technology. Students should be able to calculate linear, quadratic, cubic, and quartic regressions to model a data set. In the free-response section of the exam, students will not only be required to arrive at a solution but also explain and provide rationales for their conclusions. Students should practice providing reasons for conclusions throughout the unit in both spoken and written form and continually refine their explanations to improve precision.

AP Precalculus Course and Exam Description

Course Framework V.1 | 27

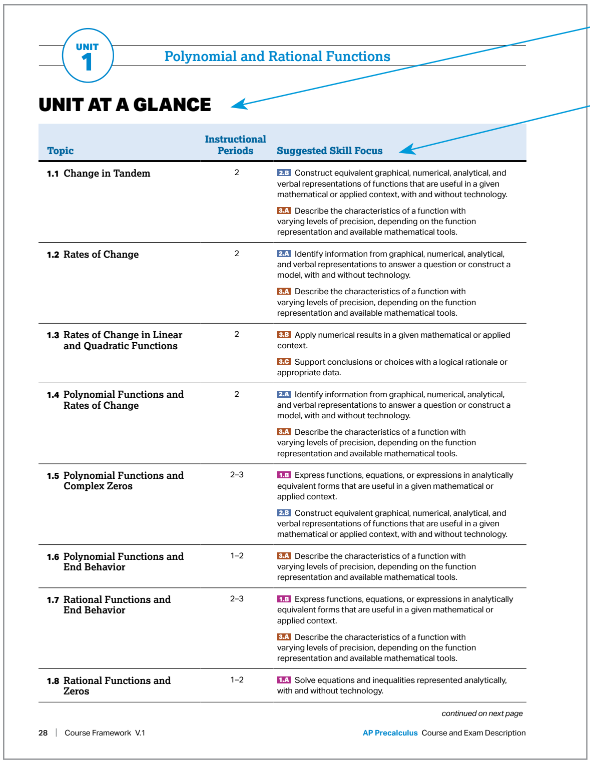
UNIT OPENERS

Developing Understanding provides an overview that contextualizes and situates the key content of the unit within the scope of the course.

The **essential questions** are thought-provoking questions that motivate students and inspire inquiry.

Building the Mathematical Practices describes specific skills within the practices that are appropriate to focus on in that unit. Certain practices have been noted to indicate areas of emphasis for that unit.

Preparing for the AP Exam provides helpful tips and common student misunderstandings identified from prior exam data.



UNIT 1 Polynomial and Rational Functions

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
1.1 Change in Tandem	2	2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. 3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.2 Rates of Change	2	2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. 3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.3 Rates of Change in Linear and Quadratic Functions	2	3.B Apply numerical results in a given mathematical or applied context. 3.C Support conclusions or choices with a logical rationale or appropriate data.
1.4 Polynomial Functions and Rates of Change	2	2.A Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. 3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.5 Polynomial Functions and Complex Zeros	2–3	1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. 2.B Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
1.6 Polynomial Functions and End Behavior	1–2	3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.7 Rational Functions and End Behavior	2–3	1.B Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. 3.A Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.8 Rational Functions and Zeros	1–2	1.A Solve equations and inequalities represented analytically, with and without technology.

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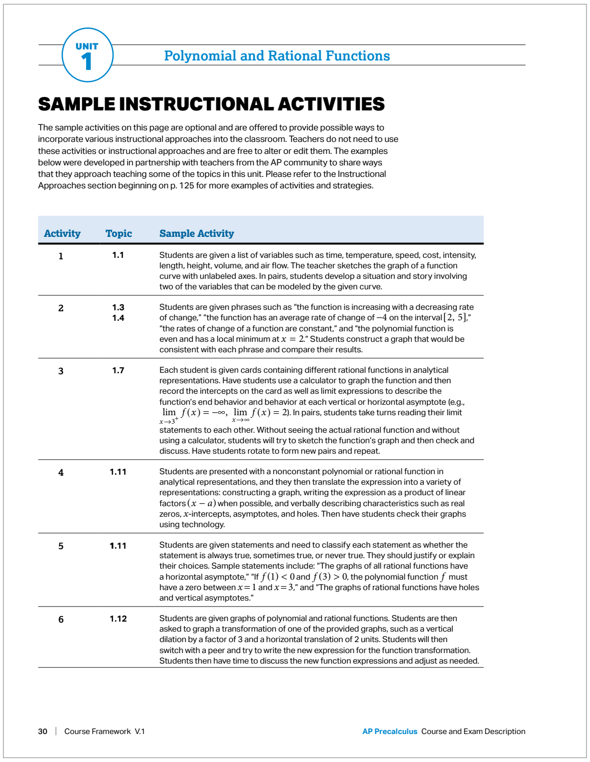
AP Precalculus Course and Exam Description

The **Unit at a Glance** table shows the topics, and suggested skills.

The **suggested skills** for each topic show possible ways to link the content in that topic to specific AP Precalculus skills. The individual skills have been thoughtfully chosen in a way that scaffolds the skills throughout the course. The questions on the Progress Checks are based on this pairing. However, AP Exam questions can pair the content with any of the skills.

Using the Unit Guides



UNIT 1

Polynomial and Rational Functions

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.1	Students are given a list of variables such as time, temperature, speed, cost, intensity, length, height, volume, and air flow. The teacher sketches the graph of a function curve with unlabeled axes. In pairs, students develop a situation and story involving two of the variables that can be modeled by the given curve.
2	1.3 1.4	Students are given phrases such as "the function is increasing with a decreasing rate of change," "the function has an average rate of change of -4 on the interval $[2, 5]$," "the rates of change of a function are constant," and "the polynomial function is even and has a local minimum at $x = 2$." Students construct a graph that would be consistent with each phrase and compare their results.
3	1.7	Each student is given cards containing different rational functions in analytical representations. Have students use a calculator to graph the function and then record the intercepts on the card as well as limit expressions to describe the function's end behavior and behavior at each vertical or horizontal asymptote (e.g., $\lim_{x \to 3^+} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 2$). In pairs, students take turns reading their limit statements to each other. Without seeing the actual rational function and without using a calculator, students will try to sketch the function's graph and then check and discuss. Have students rotate to form new pairs and repeat.
4	1.11	Students are presented with a nonconstant polynomial or rational function in analytical representations, and they then translate the expression into a variety of representations: constructing a graph, writing the expression as a product of linear factors $(x - a)$ when possible, and verbally describing characteristics such as real zeros, x-intercepts, asymptotes, and holes. Then have students check their graphs using technology.
5	1.11	Students are given statements and need to classify each statement as whether the statement is always true, sometimes true, or never true. They should justify or explain their choices. Sample statements include: "The graphs of all rational functions have a horizontal asymptote," "If $f(1) < 0$ and $f(3) > 0$, the polynomial function f must have a zero between $x=1$ and $x=3$," and "The graphs of rational functions have holes and vertical asymptotes."
6	1.12	Students are given graphs of polynomial and rational functions. Students are then asked to graph a transformation of one of the provided graphs, such as a vertical dilation by a factor of 3 and a horizontal translation of 2 units. Students will then switch with a peer and try to write the new expression for the function transformation. Students then have time to discuss the new function expressions and adjust as needed.

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AP Precalculus Course and Exam Description

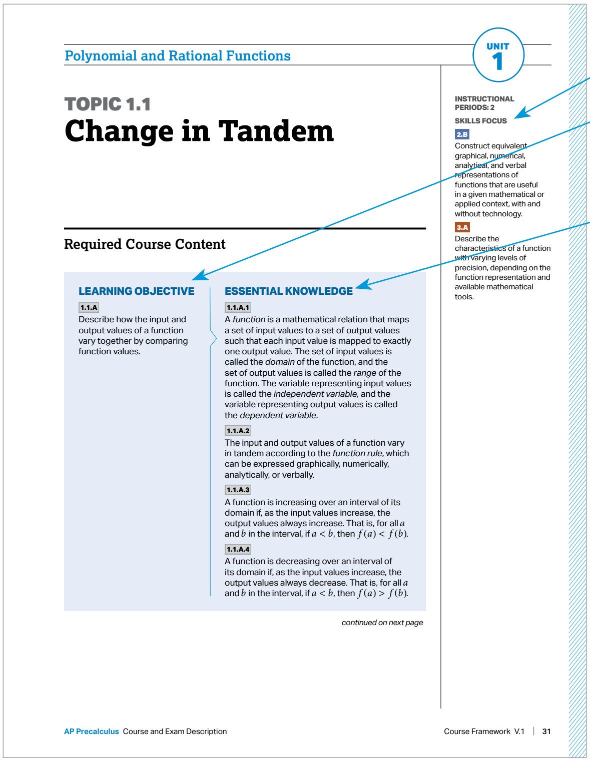
The **Sample Instructional Activities** page includes optional activities that can help teachers tie together the content and skill for a particular topic.

TOPIC PAGES

The **suggested skill** offers a possible skill to pair with the topic.

Learning objectives define what a student needs to be able to do with content knowledge in order to progress through the course.

Essential knowledge statements describe the knowledge required to perform the learning objective.



Polynomial and Rational Functions

UNIT 1

TOPIC 1.1

Change in Tandem

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.1.A Describe how the input and output values of a function vary together by comparing function values.	1.1.A.1 A *function* is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the *domain* of the function, and the set of output values is called the *range* of the function. The variable representing input values is called the *independent variable*, and the variable representing output values is called the *dependent variable*. 1.1.A.2 The input and output values of a function vary in tandem according to the *function rule*, which can be expressed graphically, numerically, analytically, or verbally. 1.1.A.3 A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$. 1.1.A.4 A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

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AP Precalculus Course and Exam Description

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AP PRECALCULUS

UNIT 1

Polynomial and Rational Functions

AP Precalculus Exam Topics
(required for college calculus placement)

30–40%
AP EXAM WEIGHTING

30–40
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 1 Part 1: Topics 1.1–1.6

Multiple-choice: 18
Free-response: 2

Progress Check Unit 1 Part 2: Topics 1.7–1.14

Multiple-choice: 24
Free-response: 2

UNIT
1

30–40% AP EXAM WEIGHTING

30–40 CLASS PERIODS

Polynomial and Rational Functions

ESSENTIAL QUESTIONS

- How do we model the intensity of light from its source?
- How can I use data and graphs to figure out the best time to purchase event tickets?
- How can we adjust known projectile motion models to account for changes in conditions?

Developing Understanding

In Unit 1, students develop understanding of two key function concepts while exploring polynomial and rational functions. The first concept is covariation, or how output values change in tandem with changing input values. The second concept is rates of change, including average rate of change, rate of change at a point, and changing rates of change. The central idea of a function as a rule for relating two simultaneously changing sets of values provides students with a vital tool that has many applications, in nature, human society, and business and industry. For example, the idea of crop yield increasing but at a decreasing rate or the efficacy of a medicine decreasing but at an increasing rate are important understandings that inform critical decisions.

Building the Mathematical Practices

3.A **3.B**

Throughout the course, students should practice communicating mathematics and developing notational fluency—and that practice should begin in Unit 1. Students should use precise language such as, “On the closed interval 0 to 1, as the value of *x* increases, the value of *y* increases then decreases.” To the fullest extent possible, students should work on functions presented in contextual scenarios such as graphs showing distance vs. time, tables showing velocity vs. time, or scenarios involving volume vs. time. In these contexts, students should use clear language when referring to variables and functions, including units of measure as appropriate. For example, when considering a problem of filling a pool with water, a student may write, “The input values of the function *V* are times in minutes, and the output values are volumes in cubic meters. The average rate of change of the function *V* over the time interval *t* equals 2 minutes to *t* equals 5 minutes is 0.4 cubic meters per minute.” Practicing communicating with precise language can help students clarify their thinking and make important connections while revealing misconceptions.

Preparing for the AP Exam

After studying Unit 1, students should be able to describe, represent, and model polynomial and rational functions and their additive and multiplicative transformations. Because part of the exam relies on technology, students should be able to identify zeros, points of intersection, and extrema using graphing calculator technology. Students should be able to calculate linear, quadratic, cubic, and quartic regressions to model a data set. In the free-response section of the exam, students will not only be required to arrive at a solution but also explain and provide rationales for their conclusions. Students should practice providing reasons for conclusions throughout the unit in both spoken and written form and continually refine their explanations to improve precision.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
1.1 Change in Tandem	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.2 Rates of Change	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.3 Rates of Change in Linear and Quadratic Functions	2	**3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.4 Polynomial Functions and Rates of Change	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.5 Polynomial Functions and Complex Zeros	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
1.6 Polynomial Functions and End Behavior	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.7 Rational Functions and End Behavior	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.8 Rational Functions and Zeros	1–2	**1.A** Solve equations and inequalities represented analytically, with and without technology.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
1.9 Rational Functions and Vertical Asymptotes	1–2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
1.10 Rational Functions and Holes	1–2	**3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.11 Equivalent Representations of Polynomial and Rational Expressions	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
1.12 Transformations of Functions	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.13 Function Model Selection and Assumption Articulation	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.14 Function Model Construction and Application	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 1.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.1	Students are given a list of variables such as time, temperature, speed, cost, intensity, length, height, volume, and air flow. The teacher sketches the graph of a function curve with unlabeled axes. In pairs, students develop a situation and story involving two of the variables that can be modeled by the given curve.
2	1.3 1.4	Students are given phrases such as "the function is increasing with a decreasing rate of change," "the function has an average rate of change of -4 on the interval $[2, 5]$," "the rates of change of a function are constant," and "the polynomial function is even and has a local minimum at $x = 2$." Students construct a graph that would be consistent with each phrase and compare their results.
3	1.7	Each student is given cards containing different rational functions in analytical representations. Have students use a calculator to graph the function and then record the intercepts on the card as well as limit expressions to describe the function's end behavior and behavior at each vertical or horizontal asymptote (e.g., $\lim_{x \to 3^+} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 2$). In pairs, students take turns reading their limit statements to each other. Without seeing the actual rational function and without using a calculator, students will try to sketch the function's graph and then check and discuss. Have students rotate to form new pairs and repeat.
4	1.11	Students are presented with a nonconstant polynomial or rational function in analytical representations, and they then translate the expression into a variety of representations: constructing a graph, writing the expression as a product of linear factors $(x - a)$ when possible, and verbally describing characteristics such as real zeros, x-intercepts, asymptotes, and holes. Then have students check their graphs using technology.
5	1.11	Students are given statements and need to classify each statement as whether the statement is always true, sometimes true, or never true. They should justify or explain their choices. Sample statements include: "The graphs of all rational functions have a horizontal asymptote," "If $f(1) < 0$ and $f(3) > 0$, the polynomial function f must have a zero between $x = 1$ and $x = 3$," and "The graphs of rational functions have holes and vertical asymptotes."
6	1.12	Students are given graphs of polynomial and rational functions. Students are then asked to graph a transformation of one of the provided graphs, such as a vertical dilation by a factor of 3 and a horizontal translation of 2 units. Students will then switch with a peer and try to write the new expression for the function transformation. Students then have time to discuss the new function expressions and adjust as needed.

TOPIC 1.1

Change in Tandem

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.1.A

Describe how the input and output values of a function vary together by comparing function values.

ESSENTIAL KNOWLEDGE

1.1.A.1

A *function* is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the *domain* of the function, and the set of output values is called the *range* of the function. The variable representing input values is called the *independent variable*, and the variable representing output values is called the *dependent variable*.

1.1.A.2

The input and output values of a function vary in tandem according to the *function rule*, which can be expressed graphically, numerically, analytically, or verbally.

1.1.A.3

A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$.

1.1.A.4

A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

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LEARNING OBJECTIVE

1.1.B

Construct a graph representing two quantities that vary with respect to each other in a contextual scenario.

ESSENTIAL KNOWLEDGE

1.1.B.1

The graph of a function displays a set of input-output pairs and shows how the values of the function's input and output values vary.

1.1.B.2

A verbal description of the way aspects of phenomena change together can be the basis for constructing a graph.

1.1.B.3

The graph of a function is *concave up* on intervals in which the rate of change is increasing.

1.1.B.4

The graph of a function is *concave down* on intervals in which the rate of change is decreasing.

1.1.B.5

The graph intersects the x-axis when the output value is zero. The corresponding input values are said to be *zeros of the function*.

TOPIC 1.2

Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.2.A Compare the rates of change at two points using average rates of change near the points.	**1.2.A.1** The average rate of change of a function over an interval of the function's domain is the constant rate of change that yields the same change in the output values as the function yielded on that interval of the function's domain. It is the ratio of the change in the output values to the change in input values over that interval. **1.2.A.2** The rate of change of a function at a point quantifies the rate at which output values would change were the input values to change at that point. The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point, if such values exist. **1.2.A.3** The rates of change at two points can be compared using average rate of change approximations over sufficiently small intervals containing each point, if such values exist.
1.2.B Describe how two quantities vary together at different points and over different intervals of a function.	**1.2.B.1** Rates of change quantify how two quantities vary together. **1.2.B.2** A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same. **1.2.B.3** A negative rate of change indicates that as one quantity increases, the other decreases.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.B
Apply numerical results in a given mathematical or applied context.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.3

Rates of Change in Linear and Quadratic Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.3.A Determine the average rates of change for sequences and functions, including linear, quadratic, and other function types.	**1.3.A.1** For a linear function, the average rate of change over any length input-value interval is constant. **1.3.A.2** For a quadratic function, the average rates of change over consecutive equal-length input-value intervals can be given by a linear function. **1.3.A.3** The average rate of change over the closed interval $[a, b]$ is the slope of the secant line from the point $(a, f(a))$ to $(b, f(b))$.
1.3.B Determine the change in the average rates of change for linear, quadratic, and other function types.	**1.3.B.1** For a linear function, since the average rates of change over consecutive equal-length input-value intervals can be given by a constant function, these average rates of change for a linear function are changing at a rate of zero. **1.3.B.2** For a quadratic function, since the average rates of change over consecutive equal-length input-value intervals can be given by a linear function, these average rates of change for a quadratic function are changing at a constant rate. **1.3.B.3** When the average rate of change over equal-length input-value intervals is increasing for all small-length intervals, the graph of the function is concave up. When the average rate of change over equal-length input-value intervals is decreasing for all small-length intervals, the graph of the function is concave down.

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TOPIC 1.4

Polynomial Functions and Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.4.A

Identify key characteristics of polynomial functions related to rates of change.

ESSENTIAL KNOWLEDGE

1.4.A.1

A nonconstant polynomial function of x is any function representation that is equivalent to the analytical form $p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, where n is a positive integer, a_i is a real number for each i from 1 to n, and a_n is nonzero. The polynomial has degree n, the leading term is a_nx^n, and the leading coefficient is a_n. A constant is also a polynomial function of degree zero.

1.4.A.2

Where a polynomial function switches between increasing and decreasing, or at the included endpoint of a polynomial with a restricted domain, the polynomial function will have a *local*, or *relative*, maximum or minimum output value. Of all local maxima, the greatest is called the *global*, or *absolute*, maximum. Likewise, the least of all local minima is called the global, or absolute, minimum.

1.4.A.3

Between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum.

1.4.A.4

Polynomial functions of an even degree will have either a global maximum or a global minimum.

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LEARNING OBJECTIVE

1.4.A

Identify key characteristics of polynomial functions related to rates of change.

ESSENTIAL KNOWLEDGE

1.4.A.5

Points of inflection of a polynomial function occur at input values where the rate of change of the function changes from increasing to decreasing or from decreasing to increasing. This occurs where the graph of a polynomial function changes from concave up to concave down or from concave down to concave up.

TOPIC 1.5

Polynomial Functions and Complex Zeros

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

1.5.A

Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.

ESSENTIAL KNOWLEDGE

1.5.A.1

If a is a complex number and $p(a) = 0$, then a is called a *zero* of the polynomial function p, or a *root* of $p(x) = 0$. If a is a real number, then $(x - a)$ is a linear factor of p if and only if a is a zero of p.

1.5.A.2

If a linear factor $(x - a)$ is repeated n times, the corresponding zero of the polynomial function has a *multiplicity* n. A polynomial function of *degree* n has exactly n complex zeros when counting multiplicities.

1.5.A.3

If a is a real zero of a polynomial function p, then the graph of $y = p(x)$ has an x-intercept at the point $(a, 0)$. Consequently, real zeros of a polynomial can be endpoints for intervals satisfying polynomial inequalities.

1.5.A.4

If $a + bi$ is a non-real zero of a polynomial function p, then its *conjugate* $a - bi$ is also a zero of p.

1.5.A.5

If the real zero, a, of a polynomial function has even multiplicity, then the signs of the output values are the same for input values near $x = a$. For these polynomial functions, the graph will be tangent to the x-axis at $x = a$.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.5.A Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.	**1.5.A.6** The degree of a polynomial function can be found by examining the successive differences of the output values over equal-interval input values. The degree of the polynomial function is equal to the least value n for which the successive $n\text{th}$ differences are constant.
1.5.B Determine if a polynomial function is even or odd.	**1.5.B.1** An *even* function is graphically symmetric over the line $x = 0$ and analytically has the property $f(-x) = f(x)$. If n is even, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an even function. **1.5.B.2** An *odd* function is graphically symmetric about the point $(0,0)$ and analytically has the property $f(-x) = -f(x)$. If n is odd, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an odd function.

TOPIC 1.6

Polynomial Functions and End Behavior

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.6.A

Describe end behaviors of polynomial functions.

ESSENTIAL KNOWLEDGE

1.6.A.1

As input values of a nonconstant polynomial function increase without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x\to\infty} p(x) = \infty$ or $\lim_{x\to\infty} p(x) = -\infty$.

1.6.A.2

As input values of a nonconstant polynomial function decrease without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim_{x\to-\infty} p(x) = \infty$ or $\lim_{x\to-\infty} p(x) = -\infty$.

1.6.A.3

The degree and sign of the leading term of a polynomial determines the end behavior of the polynomial function, because as the input values increase or decrease without bound, the values of the leading term dominate the values of all lower-degree terms.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 1.7

Rational Functions and End Behavior

Required Course Content

LEARNING OBJECTIVE

1.7.A
Describe end behaviors of rational functions.

ESSENTIAL KNOWLEDGE

1.7.A.1
A rational function is analytically represented as a quotient of two polynomial functions and gives a measure of the relative size of the polynomial function in the numerator compared to the polynomial function in the denominator for each value in the rational function's domain.

1.7.A.2
The end behavior of a rational function will be affected most by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms.

1.7.A.3
If the polynomial in the numerator dominates the polynomial in the denominator for input values of large magnitude, then the quotient of the leading terms is a nonconstant polynomial, and the original rational function has the end behavior of that polynomial. If that polynomial is linear, then the graph of the rational function has a slant asymptote parallel to the graph of the line.

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LEARNING OBJECTIVE

1.7.A

Describe end behaviors of rational functions.

ESSENTIAL KNOWLEDGE

1.7.A.4

If neither polynomial in a rational function dominates the other for input values of large magnitude, then the quotient of the leading terms is a constant, and that constant indicates the location of a horizontal asymptote of the graph of the original rational function.

1.7.A.5

If the polynomial in the denominator dominates the polynomial in the numerator for input values of large magnitude, then the quotient of the leading terms is a rational function with a constant in the numerator and nonconstant polynomial in the denominator, and the graph of the original rational function has a horizontal asymptote at $y = 0$.

1.7.A.6

When the graph of a rational function r has a horizontal asymptote at $y = b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to \infty} r(x) = b$ or $\lim_{x \to -\infty} r(x) = b$.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

1.A

Solve equations and inequalities represented analytically, with and without technology.

TOPIC 1.8

Rational Functions and Zeros

Required Course Content

LEARNING OBJECTIVE

1.8.A

Determine the zeros of rational functions.

ESSENTIAL KNOWLEDGE

1.8.A.1

The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain.

1.8.A.2

The real zeros of both polynomial functions of a rational function r are endpoints or asymptotes for intervals satisfying the rational function inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

TOPIC 1.9

Rational Functions and Vertical Asymptotes

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

Required Course Content

LEARNING OBJECTIVE

1.9.A

Determine vertical asymptotes of graphs of rational functions.

ESSENTIAL KNOWLEDGE

1.9.A.1

If the value a is a real zero of the polynomial function in the denominator of a rational function and is not also a real zero of the polynomial function in the numerator, then the graph of the rational function has a vertical asymptote at $x = a$. Furthermore, a vertical asymptote also occurs at $x = a$ if the multiplicity of a as a real zero in the denominator is greater than its multiplicity as a real zero in the numerator.

1.9.A.2

Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial function in the denominator are arbitrarily close to zero, so the values of the rational function r increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to a^+} r(x) = \infty$ or $\lim_{x \to a^+} r(x) = -\infty$ for input values near a and greater than a, and $\lim_{x \to a^-} r(x) = \infty$ or $\lim_{x \to a^-} r(x) = -\infty$ for input values near a and less than a.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.10

Rational Functions and Holes

Required Course Content

LEARNING OBJECTIVE

1.10.A

Determine holes in graphs of rational functions.

ESSENTIAL KNOWLEDGE

1.10.A.1

If the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator, then the graph of the rational function has a hole at the corresponding input value.

1.10.A.2

If the graph of a rational function r has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values sufficiently close to c. If input values sufficiently close to c correspond to output values arbitrarily close to L, then the hole is located at the point with coordinates (c, L). The corresponding mathematical notation is $\lim_{x \to c} r(x) = L$. It should be noted that $\lim_{x \to c^-} r(x) = \lim_{x \to c^+} r(x) = \lim_{x \to c} r(x) = L$.

TOPIC 1.11

Equivalent Representations of Polynomial and Rational Expressions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.11.A Rewrite polynomial and rational expressions in equivalent forms.	**1.11.A.1** Because the factored form of a polynomial or rational function readily provides information about real zeros, it can reveal information about x-intercepts, asymptotes, holes, domain, and range. **1.11.A.2** The standard form of a polynomial or rational function can reveal information about end behaviors of the function. **1.11.A.3** The information extracted from different analytic representations of the same polynomial or rational function can be used to answer questions in context.
1.11.B Determine the quotient of two polynomial functions using long division.	**1.11.B.1** Polynomial long division is an algebraic process similar to numerical long division involving a quotient and remainder. If the polynomial f is divided by the polynomial g, then f can be rewritten as $f(x) = g(x)q(x) + r(x)$, where q is the quotient, r is the remainder, and the degree of r is less than the degree of g. **1.11.B.2** The result of polynomial long division is helpful in finding equations of slant asymptotes for graphs of rational functions.

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LEARNING OBJECTIVE

1.11.C

Rewrite the repeated product of binomials using the binomial theorem.

ESSENTIAL KNOWLEDGE

1.11.C.1

The binomial theorem utilizes the entries in a single row of Pascal's Triangle to more easily expand expressions of the form $(a + b)^n$, including polynomial functions of the form $p(x) = (x + c)^n$, where c is a constant.

TOPIC 1.12

Transformations of Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.12.A

Construct a function that is an additive and/or multiplicative transformation of another function.

ESSENTIAL KNOWLEDGE

1.12.A.1

The function $g(x) = f(x) + k$ is an additive transformation of the function f that results in a vertical translation of the graph of f by k units.

1.12.A.2

The function $g(x) = f(x + h)$ is an additive transformation of the function f that results in a horizontal translation of the graph of f by $-h$ units.

1.12.A.3

The function $g(x) = af(x)$, where $a \neq 0$, is a multiplicative transformation of the function f that results in a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

1.12.A.4

The function $g(x) = f(bx)$, where $b \neq 0$, is a multiplicative transformation of the function f that results in a horizontal dilation of the graph of f by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

1.12.A.5

Additive and multiplicative transformations can be combined, resulting in combinations of horizontal and vertical translations and dilations.

1.12.A.6

The domain and range of a function that is a transformation of a parent function may be different from those of the parent function.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.13

Function Model Selection and Assumption Articulation

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.13.A Identify an appropriate function type to construct a function model for a given scenario.	**1.13.A.1** Linear functions model data sets or aspects of contextual scenarios that demonstrate roughly constant rates of change. **1.13.A.2** Quadratic functions model data sets or aspects of contextual scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value. **1.13.A.3** Geometric contexts involving area or two dimensions can often be modeled by quadratic functions. Geometric contexts involving volume or three dimensions can often be modeled by cubic functions. **1.13.A.4** Polynomial functions model data sets or contextual scenarios with multiple real zeros or multiple maxima or minima. **1.13.A.5** A polynomial function of degree n models data sets or contextual scenarios that demonstrate roughly constant nonzero nth differences. **1.13.A.6** A polynomial function of degree n or less can be used to model a graph of $n+1$ points with distinct input values.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.13.A Identify an appropriate function type to construct a function model for a given scenario.	**1.13.A.7** A piecewise-defined function consists of a set of functions defined over nonoverlapping domain intervals and is useful for modeling a data set or contextual scenario that demonstrates different characteristics over different intervals.
1.13.B Describe assumptions and restrictions related to building a function model.	**1.13.B.1** A model may have underlying assumptions about what is consistent in the model. **1.13.B.2** A model may have underlying assumptions about how quantities change together. **1.13.B.3** A model may require domain restrictions based on mathematical clues, contextual clues, or extreme values in the data set. **1.13.B.4** A model may require range restrictions, such as rounding values, based on mathematical clues, contextual clues, or extreme values in the data set.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 1.14
Function Model Construction and Application

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.14.A Construct a linear, quadratic, cubic, quartic, polynomial of degree n, or related piecewise-defined function model.	**1.14.A.1** A model can be constructed based on restrictions identified in a mathematical or contextual scenario. **1.14.A.2** A model of a data set or a contextual scenario can be constructed using transformations of the parent function. **1.14.A.3** A model of a data set can be constructed using technology and regressions, including linear, quadratic, cubic, and quartic regressions. **1.14.A.4** A piecewise-defined function model can be constructed through a combination of modeling techniques.
1.14.B Construct a rational function model based on a context.	**1.14.B.1** Data sets and aspects of contextual scenarios involving quantities that are inversely proportional can often be modeled by rational functions. For example, the magnitudes of both gravitational force and electromagnetic force between objects are inversely proportional to the objects' squared distance.
1.14.C Apply a function model to answer questions about a data set or contextual scenario.	**1.14.C.1** A model can be used to draw conclusions about the modeled data set or contextual scenario, including answering key questions and predicting values, rates of change, average rates of change, and changing rates of change. Appropriate units of measure should be extracted or inferred from the given context.

AP PRECALCULUS

UNIT 2

Exponential and Logarithmic Functions

AP Precalculus Exam Topics
(required for college calculus placement)

27–40%
AP EXAM WEIGHTING

30–45
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 2 Part 1: Topics 2.1–2.8

Multiple-choice: 24
Free-response: 2

Progress Check Unit 2 Part 2: Topics 2.9–2.15

Multiple-choice: 24
Free-response: 2

UNIT
2

27–40% AP EXAM WEIGHTING

30–45 CLASS PERIODS

Exponential and Logarithmic Functions

Developing Understanding

In Unit 2, students build an understanding of exponential and logarithmic functions. Exponential and logarithmic function models are widespread in the natural and social sciences. When an aspect of a phenomenon changes proportionally to the existing amount, exponential and logarithmic models are employed to harness the information. Exponential functions are key to modeling population growth, radioactive decay, interest rates, and the amount of medication in a patient. Logarithmic functions are useful in modeling sound intensity and frequency, the magnitude of earthquakes, the pH scale in chemistry, and the working memory in humans. The study of these two function types touches careers in business, medicine, chemistry, physics, education, and human geography, among others.

ESSENTIAL QUESTIONS

- How can I make a single model that merges the interest I earn from my bank with the taxes that are due so I can know how much I will have in the end?
- How can we adjust the scale of distance for a model of planets in the solar system so the relationships among the planets are easier to see?
- If different functions can be used to model data, how do we pick which one is best?

Building the Mathematical Practices

2.A **2.B**

Students should learn to communicate differences and similarities among arithmetic sequences, linear functions, geometric sequences, and exponential functions. Students can develop a deeper understanding of these four function types by considering how each would be represented in a graph, in a table, in an analytical representation, and through verbal descriptions of related scenarios. Examining multiple representations is also powerful in understanding composition of functions and relationships between functions and their inverse functions. In this unit, multiple representations should be used to explore the inverse relationship between exponential and logarithmic functions.

Preparing for the AP Exam

Students should practice clearly delineating the processes that lead to answers when solving equations, finding equivalent expressions, and building function models. Since manipulating exponential and logarithmic functions requires a great deal of precision, students should avoid "skipping steps" as errors can be easily introduced with these function types. Furthermore, on the free-response section of the AP Exam, answers without supporting work may not be acceptable, so students should practice clear communication throughout the course. Students will continue to use technology to explore multiple representations in this unit as they did in Unit 1 and will expand their use of calculating regressions to include exponential and logarithmic regressions. They should practice calculating various regressions on data sets, plot the residuals, and justify the choice of a function model based on analysis of the residuals. Students should be fluent with the practices of entering data, calculating regressions, and plotting residuals with the graphing calculator before the AP Exam.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
2.1 Change in Arithmetic and Geometric Sequences	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.2 Change in Linear and Exponential Functions	2	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.3 Exponential Functions	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.4 Exponential Function Manipulation	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.5 Exponential Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.6 Competing Function Model Validation	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
2.7 Composition of Functions	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
2.8 Inverse Functions	2–3	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
2.9 Logarithmic Expressions	1–2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
2.10 Inverses of Exponential Functions	2	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
2.11 Logarithmic Functions	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.12 Logarithmic Function Manipulation	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.13 Exponential and Logarithmic Equations and Inequalities	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.
2.14 Logarithmic Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.15 Semi-log Plots	2–3	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 2.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	2.2	Five students randomly choose different integers between -6 and 6 that will be used as input values for a function. In groups, students determine appropriate output values for the five input values that illustrate a linear function relationship, then those that illustrate an exponential function relationship. The groups defend their decisions to the rest of the class by explaining how they know their values represent linear and exponential functions.
2	2.2	Have students pair up and attempt to walk (in a straight line from a specific starting point) using the following walking patterns: i) at a constant rate of change of 2 feet per second based on the model $d = 2t$; ii) at an increasing rate of change based on the model $d = 2^t$. Ask each pair to describe the differences in their two experiences. Follow up with applied questions such as "How far would you walk in 10 seconds? in 33 seconds?" or "Using each model, how long would it take you to travel around Earth (~25,000 miles)?, travel to the Moon (~239,000 miles)?"
3	2.5	Student *A* and Student *B* are both provided with a context that can be modeled using exponential functions such as doubling or halving a certain substance. Without looking at each other's work, each student generates a graph and an expression to model the exponential function. Then, the students switch papers to review each other's solutions.
4	2.6	Students are given multiple tables of data, each of which is well-modeled by a linear, exponential, or quadratic function. In pairs, students calculate all three regressions, observe the graphs of the regressions, observe the corresponding residual plots, and draw conclusions about the relationships between the graphs and the residuals.
5	2.11	Students are given a sheet of paper containing several logarithmic functions represented analytically and their corresponding graphs, in mixed order, in a separate column. In small groups, students match the corresponding representations and take turns explaining how they know each is matched correctly.
6	2.13	Place 10 to 16 index cards around the room. The top of each card contains a problem, and the bottom of each card contains a solution to a problem from a different card. Students will start at a card, solve the problem, and then search for the next card with their solution. The cards should be set up such that the students will find the matches of all of the cards.

TOPIC 2.1

Change in Arithmetic and Geometric Sequences

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.1.A Express arithmetic sequences found in mathematical and contextual scenarios as functions of the whole numbers.	**2.1.A.1** A *sequence* is a function from the whole numbers to the real numbers. Consequently, the graph of a sequence consists of discrete points instead of a curve. **2.1.A.2** Successive terms in an arithmetic sequence have a common difference, or constant rate of change. **2.1.A.3** The general term of an arithmetic sequence with a common difference d is denoted by a_n and is given by $a_n = a_0 + dn$, where a_0 is the initial value, or by $a_n = a_k + d(n-k)$, where a_k is the kth term of the sequence.
2.1.B Express geometric sequences found in mathematical and contextual scenarios as functions of the whole numbers.	**2.1.B.1** Successive terms in a geometric sequence have a common ratio, or constant proportional change. **2.1.B.2** The general term of a geometric sequence with a common ratio r is denoted by g_n and is given by $g_n = g_0 r^n$, where g_0 is the initial value, or by $g_n = g_k r^{(n-k)}$, where g_k is the kth term of the sequence. **2.1.B.3** Increasing arithmetic sequences increase equally with each step, whereas increasing geometric sequences increase by a larger amount with each successive step.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

TOPIC 2.2

Change in Linear and Exponential Functions

Required Course Content

LEARNING OBJECTIVE

2.2.A

Construct functions of the real numbers that are comparable to arithmetic and geometric sequences.

ESSENTIAL KNOWLEDGE

2.2.A.1

Linear functions of the form $f(x) = b + mx$ are similar to arithmetic sequences of the form $a_n = a_0 + dn$, as both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d).

2.2.A.2

Similar to arithmetic sequences of the form $a_n = a_k + d(n - k)$, which are based on a known difference, d, and a kth term, linear functions can be expressed in the form $f(x) = y_i + m(x - x_i)$ based on a known slope, m, and a point, (x_i, y_i).

2.2.A.3

Exponential functions of the form $f(x) = ab^x$ are similar to geometric sequences of the form $g_n = g_0 r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant proportion (b or r).

2.2.A.4

Similar to geometric sequences of the form $g_n = g_k r^{(n-k)}$, which are based on a known ratio, r, and a kth term, exponential functions can be expressed in the form $f(x) = y_i r^{(x - x_i)}$ based on a known ratio, r, and a point, (x_i, y_i).

2.2.A.5

Sequences and their corresponding functions may have different domains.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.2.B Describe similarities and differences between linear and exponential functions.	**2.2.B.1** Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential. **2.2.B.2** Linear functions of the form $f(x) = b + mx$ and exponential functions of the form $f(x) = ab^x$ can both be expressed analytically in terms of an initial value and a constant involved with change. However, linear functions are based on addition, while exponential functions are based on multiplication. **2.2.B.3** Arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the property that they can be determined by two distinct sequence or function values.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.3

Exponential Functions

Required Course Content

LEARNING OBJECTIVE

2.3.A
Identify key characteristics of exponential functions.

ESSENTIAL KNOWLEDGE

2.3.A.1
The general form of an exponential function is $f(x) = ab^x$, with the *initial value* a, where $a \neq 0$, and the *base* b, where $b > 0$, and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is said to demonstrate *exponential growth*. When $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate *exponential decay*.

2.3.A.2
When the natural numbers are input values in an exponential function, the input value specifies the number of factors of the base to be applied to the function's initial value. The domain of an exponential function is all real numbers.

2.3.A.3
Because the output values of exponential functions in general form are proportional over equal-length input-value intervals, exponential functions are always increasing or always decreasing, and their graphs are always concave up or always concave down. Consequently, exponential functions do not have extrema except on a closed interval, and their graphs do not have points of inflection.

2.3.A.4
If the values of the additive transformation function $g(x) = f(x) + k$ of any function f are proportional over equal-length input-value intervals, then f is exponential.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.3.A Identify key characteristics of exponential functions.	**2.3.A.5** For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $\lim_{x \to \pm\infty} ab^x = \infty$, $\lim_{x \to \pm\infty} ab^x = -\infty$, or $\lim_{x \to \pm\infty} ab^x = 0$.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.4

Exponential Function Manipulation

Required Course Content

LEARNING OBJECTIVE

2.4.A

Rewrite exponential expressions in equivalent forms.

ESSENTIAL KNOWLEDGE

2.4.A.1

The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.

2.4.A.2

The power property for exponents states that $\left(b^m\right)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$, is equivalent to a change of the base of an exponential function, $f(x) = \left(b^c\right)^x$, where b^c is a constant and $c \neq 0$.

2.4.A.3

The negative exponent property states that $b^{-n} = \frac{1}{b^n}$.

2.4.A.4

The value of an exponential expression involving an exponential unit fraction, such as $b^{(1/k)}$ where k is a natural number, is the kth root of b, when it exists.

TOPIC 2.5

Exponential Function Context and Data Modeling

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

2.5.A

Construct a model for situations involving proportional output values over equal-length input-value intervals.

ESSENTIAL KNOWLEDGE

2.5.A.1

Exponential functions model growth patterns where successive output values over equal-length input-value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.

2.5.A.2

A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.

2.5.A.3

An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.

2.5.A.4

Exponential function models can be constructed by applying transformations to $f(x) = ab^x$ based on characteristics of a contextual scenario or data set.

2.5.A.5

Exponential function models can be constructed for a data set with technology using exponential regressions.

2.5.A.6

The natural base e, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios.

LEARNING OBJECTIVE

2.5.B

Apply exponential models to answer questions about a data set or contextual scenario.

ESSENTIAL KNOWLEDGE

2.5.B.1

For an exponential model in general form $f(x) = ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context.

2.5.B.2

Equivalent forms of an exponential function can reveal different properties of the function. For example, if d represents number of days, then the base of $f(d) = 2^d$ indicates that the quantity increases by a factor of 2 every day, but the equivalent form $f(d) = \left(2^7\right)^{(d/7)}$ indicates that the quantity increases by a factor of 2^7 every week.

2.5.B.3

Exponential models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain.

TOPIC 2.6

Competing Function Model Validation

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.6.A Construct linear, quadratic, and exponential models based on a data set.	**2.6.A.1** Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models. **2.6.A.2** Models can be compared based on contextual clues and applicability to determine which model is most appropriate.
2.6.B Validate a model constructed from a data set.	**2.6.B.1** A model is justified as *appropriate* for a data set if the graph of the residuals of a regression, the residual plot, appear without pattern. **2.6.B.2** The difference between the predicted and actual values is the *error* in the model. Depending on the data set and context, it may be more appropriate to have an underestimate or overestimate for any given interval.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 2.7

Composition of Functions

Required Course Content

LEARNING OBJECTIVE

2.7.A

Evaluate the composition of two or more functions for given values.

ESSENTIAL KNOWLEDGE

2.7.A.1

If f and g are functions, the composite function $f \circ g$ maps a set of input values to a set of output values such that the output values of g are used as input values of f. For this reason, the domain of the composite function is restricted to those input values of g for which the corresponding output value is in the domain of f. $(f \circ g)(x)$ can also be represented as $f(g(x))$.

2.7.A.2

Values for the composite function $f \circ g$ can be calculated or estimated from the graphical, numerical, analytical, or verbal representations of f and g by using output values from g as input values for f.

2.7.A.3

The composition of functions is not commutative; that is, $f \circ g$ and $g \circ f$ are typically different functions; therefore, $f(g(x))$ and $g(f(x))$ are typically different values.

2.7.A.4

If the function $f(x) = x$ is composed with any function g, the resulting composite function is the same as g; that is, $g(f(x)) = f(g(x)) = g(x)$. The function $f(x) = x$ is called the *identity function*. When composing two functions, the identify function acts in the same way as 0, the additive identity, when adding two numbers and 1, the multiplicative identity, when multiplying two numbers.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.7.B Construct a representation of the composition of two or more functions.	**2.7.B.1** Function composition is useful for relating two quantities that are not directly related by an existing formula. **2.7.B.2** When analytic representations of the functions f and g are available, an analytic representation of $f(g(x))$ can be constructed by substituting $g(x)$ for every instance of x in f. **2.7.B.3** A numerical or graphical representation of $f \circ g$ can often be constructed by calculating or estimating values for $(x, f(g(x)))$.
2.7.C Rewrite a given function as a composition of two or more functions.	**2.7.C.1** Functions given analytically can often be decomposed into less complicated functions. When properly decomposed, the variable in one function should replace each instance of the function with which it was composed. **2.7.C.2** An additive transformation of a function, f, that results in vertical and horizontal translations can be understood as the composition of $g(x) = x + k$ with f. **2.7.C.3** A multiplicative transformation of a function, f, that results in vertical and horizontal dilations can be understood as the composition of $g(x) = kx$ with f.

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INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.A

Solve equations and inequalities represented analytically, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 2.8

Inverse Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.8.A Determine the input-output pairs of the inverse of a function.	**2.8.A.1** On a specified domain, a function, f, has an inverse function, or is invertible, if each output value of f is mapped from a unique input value. The domain of a function may be restricted in many ways to make the function invertible. **2.8.A.2** An inverse function can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a) = b$, then $f^{-1}(b) = a$. Alternately, on its invertible domain, if a function consists of input-output pairs (a, b), then the inverse function consists of input-output pairs (b, a).
2.8.B Determine the inverse of a function on an invertible domain.	**2.8.B.1** The composition of a function, f, and its inverse function, f^{-1}, is the identity function; that is, $f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$. **2.8.B.2** On a function's invertible domain, the function's range and domain are the inverse function's domain and range, respectively. The inverse of the table of values of $y = f(x)$ can be found by reversing the input-output pairs; that is, (a, b) corresponds to (b, a).

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LEARNING OBJECTIVE

2.8.B

Determine the inverse of a function on an invertible domain.

ESSENTIAL KNOWLEDGE

2.8.B.3

The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y-axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x) = x$.

2.8.B.4

The inverse of the function can be found by determining the inverse operations to reverse the mapping. One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y = f(x)$, then solving for $y = f^{-1}(x)$.

2.8.B.5

In addition to limiting the domain of a function to obtain an inverse function, contextual restrictions may also limit the applicability of an inverse function.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

TOPIC 2.9

Logarithmic Expressions

Required Course Content

LEARNING OBJECTIVE

2.9.A

Evaluate logarithmic expressions.

ESSENTIAL KNOWLEDGE

2.9.A.1

The logarithmic expression $\log_b c$ is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c. That is, $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$. (when the base of a logarithmic expression is not specified, it is understood as the common logarithm with a base of 10)

2.9.A.2

The values of some logarithmic expressions are readily accessible through basic arithmetic while other values can be estimated through the use of technology.

2.9.A.3

On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be $0,\ 1,\ 2,\ \ldots$, while on a logarithmic scale, using logarithm base 10, the units might be $10^0,\ 10^1,\ 10^2,\ \ldots$.

TOPIC 2.10

Inverses of Exponential Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

2.10.A

Construct representations of the inverse of an exponential function with an initial value of 1.

ESSENTIAL KNOWLEDGE

2.10.A.1

The general form of a logarithmic function is $f(x) = a\log_b x$, with *base* b, where $b > 0, b \neq 1$, and $a \neq 0$.

2.10.A.2

The way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Output values of general-form exponential functions change proportionately as input values increase in equal-length intervals. However, input values of general-form logarithmic functions change proportionately as output values increase in equal-length intervals. Alternately, exponential growth is characterized by output values changing multiplicatively as input values change additively, whereas logarithmic growth is characterized by output values changing additively as input values change multiplicatively.

2.10.A.3

$f(x) = \log_b x$ and $g(x) = b^x$, where $b > 0$ and $b \neq 1$, are inverse functions. That is, $g(f(x)) = f(g(x)) = x$.

2.10.A.4

The graph of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, is a reflection of the graph of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, over the graph of the identity function $h(x) = x$.

2.10.A.5

If (s, t) is an ordered pair of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, then (t, s) is an ordered pair of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.11

Logarithmic Functions

Required Course Content

LEARNING OBJECTIVE

2.11.A

Identify key characteristics of logarithmic functions.

ESSENTIAL KNOWLEDGE

2.11.A.1

The domain of a logarithmic function in general form is any real number greater than zero, and its range is all real numbers.

2.11.A.2

Because logarithmic functions are inverses of exponential functions, logarithmic functions are also always increasing or always decreasing, and their graphs are either always concave up or always concave down. Consequently, logarithmic functions do not have extrema except on a closed interval, and their graphs do not have points of inflection.

2.11.A.3

The additive transformation function $g(x) = f(x + k)$, where $k \neq 0$, of a logarithmic function f in general form does not have input values that are proportional over equal-length output-value intervals. However, if the input values of the additive transformation function, $g(x) = f(x + k)$, of any function f are proportional over equal-length output value intervals, then f is logarithmic.

2.11.A.4

With their limited domain, logarithmic functions in general form are vertically asymptotic to $x = 0$, with an end behavior that is unbounded. That is, for a logarithmic function in general form, $\lim\limits_{x\to 0^+} a\log_b x = \pm\infty$ and $\lim\limits_{x\to\infty} a\log_b x = \pm\infty$.

TOPIC 2.12

Logarithmic Function Manipulation

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

2.12.A

Rewrite logarithmic expressions in equivalent forms.

ESSENTIAL KNOWLEDGE

2.12.A.1

The product property for logarithms states that $\log_b(xy) = \log_b x + \log_b y$. Graphically, this property implies that every horizontal dilation of a logarithmic function, $f(x) = \log_b(kx)$, is equivalent to a vertical translation, $f(x) = \log_b(kx) = \log_b k + \log_b x = a + \log_b x$, where $a = \log_b k$.

2.12.A.2

The power property for logarithms states that $\log_b x^n = n\log_b x$. Graphically, this property implies that raising the input of a logarithmic function to a power, $f(x) = \log_b x^k$, results in a vertical dilation, $f(x) = \log_b x^k = k\log_b x$.

2.12.A.3

The change of base property for logarithms states that $\log_b x = \dfrac{\log_a x}{\log_a b}$, where $a > 0$ and $a \neq 1$. This implies that all logarithmic functions are vertical dilations of each other.

2.12.A.4

The function $f(x) = \ln x$ is a logarithmic function with the natural base e; that is, $\ln x = \log_e x$.

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

TOPIC 2.13

Exponential and Logarithmic Equations and Inequalities

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.13.A Solve exponential and logarithmic equations and inequalities.	**2.13.A.1** Properties of exponents, properties of logarithms, and the inverse relationship between exponential and logarithmic functions can be used to solve equations and inequalities involving exponents and logarithms. **2.13.A.2** When solving exponential and logarithmic equations found through analytical or graphical methods, the results should be examined for extraneous solutions precluded by the mathematical or contextual limitations. **2.13.A.3** Logarithms can be used to rewrite expressions involving exponential functions in different ways that may reveal helpful information. Specifically, $b^x = c^{(\log_c b)(x)}$.
2.13.B Construct the inverse function for exponential and logarithmic functions.	**2.13.B.1** The function $f(x) = ab^{(x-h)} + k$ is a combination of additive transformations of an exponential function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping. **2.13.B.2** The function $f(x) = a\log_b(x-h) + k$ is a combination of additive transformations of a logarithmic function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping.

TOPIC 2.14

Logarithmic Function Context and Data Modeling

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

2.14.A

Construct a logarithmic function model.

ESSENTIAL KNOWLEDGE

2.14.A.1

Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal-length output-value intervals. Alternately, if the output value is a whole number, it indicates how many times the initial value has been multiplied by the proportion.

2.14.A.2

A logarithmic function model can be constructed from an appropriate proportion and a real zero or from two input-output pairs.

2.14.A.3

Logarithmic function models can be constructed by applying transformations to $f(x) = a\log_b x$ based on characteristics of a context or data set.

2.14.A.4

Logarithmic function models can be constructed for a data set with technology using logarithmic regressions.

2.14.A.5

The natural logarithm function is often useful in modeling real-world phenomena.

2.14.A.6

Logarithmic function models can be used to predict values for the dependent variable.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 2.15

Semi-log Plots

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.15.A Determine if an exponential model is appropriate by examining a semi-log plot of a data set.	**2.15.A.1** In a semi-log plot, one of the axes is logarithmically scaled. When the y-axis of a semi-log plot is logarithmically scaled, data or functions that demonstrate exponential characteristics will appear linear. **2.15.A.2** An advantage of semi-log plots is that a constant never needs to be added to the dependent variable values to reveal that an exponential model is appropriate.
2.15.B Construct the linearization of exponential data.	**2.15.B.1** Techniques used to model linear functions can be applied to a semi-log graph. **2.15.B.2** For an exponential model of the form $y = ab^x$, the corresponding linear model for the semi-log plot is $y = (\log_n b)x + \log_n a$, where $n > 0$ and $n \neq 1$. Specifically, the linear rate of change is $\log_n b$, and the initial linear value is $\log_n a$.

AP PRECALCULUS

UNIT 3

Trigonometric and Polar Functions

AP Precalculus Exam Topics
(required for college calculus placement)

30–35%
AP EXAM WEIGHTING

35–50
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 3 Part 1: Topics 3.1–3.7

Multiple-choice: 21
Free-response: 2

Progress Check Unit 3 Part 2: Topics 3.8–3.15

Multiple-choice: 24
Free-response: 2

UNIT 3

30–35% AP EXAM WEIGHTING | **35–50** CLASS PERIODS

Trigonometric and Polar Functions

ESSENTIAL QUESTIONS

- Since energy usage goes up and down through the year, how can I use trends in data to predict my monthly electricity bills when I get my first apartment?
- How do we model aspects of circular and spinning objects without using complex equations from the *x*-*y* rectangular-based coordinate system?
- How does right triangle trigonometry from geometry relate to trigonometric functions?

Developing Understanding

In Unit 3, students explore trigonometric functions and their relation to the angles and arcs of a circle. Since their output values repeat with every full revolution around the circle, trigonometric functions are ideal for modeling periodic, or repeated pattern phenomena, such as: the highs and lows of a wave, the blood pressure produced by a heart, and the angle from the North Pole to the Sun year to year. Furthermore, periodicity is found in human inventions and social phenomena. For example, moving parts of an analog clock are modeled by a trigonometric function with respect to each other or with respect to time; traffic flow at an intersection over the course of a week demonstrates daily periodicity; and demand for a particular product over the course of a year falls into an annually repeating pattern. Polar functions, which are also explored in this unit, have deep ties to trigonometric functions as they are both based on the circle. Polar functions are defined on the polar coordinate system that uses the circular concepts of radii and angles to describe location instead of rectangular concepts of left-right and up-down, which students have worked with previously. Trigonometry serves as the bridge between the two systems.

Building the Mathematical Practices

1.A 1.B 1.C 2.B 3.A

Students should have multiple experiences transitioning among, and communicating about, the various representations of trigonometric functions, especially sinusoidal functions. It is important that, in addition to solving trigonometric equations and finding equivalent trigonometric expressions, students build sinusoidal models with and without technology and practice constructing different representations. As students transition to thinking in the polar plane, they will refine their communications related to characteristics of functions. The more casual language that students may have adopted such as “goes up” and “goes down” will need to be replaced with more careful language that addresses a function’s behavior related to angles and radii.

Preparing for the AP Exam

As sinusoidal function content pairs well with each of the mathematical practices and skills in the course, students should practice applying each skill to each trigonometric function objective. On the AP Exam, students will need to describe characteristics of sinusoidal functions, such as amplitude, vertical shift, period, and phase shift, give reasons for why chosen values are consistent with information provided, calculate sinusoidal regressions using a graphing calculator, and apply trigonometric function models to contexts. Students should be practicing these skills in daily activities, so they are able to use them fluently on the AP Exam.

UNIT 3

Trigonometric and Polar Functions

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
3.1 Periodic Phenomena	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.2 Sine, Cosine, and Tangent	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.3 Sine and Cosine Function Values	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
3.4 Sine and Cosine Function Graphs	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.5 Sinusoidal Functions	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.6 Sinusoidal Function Transformations	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.7 Sinusoidal Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
3.8 The Tangent Function	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
3.9 Inverse Trigonometric Functions	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.10 Trigonometric Equations and Inequalities	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
3.11 The Secant, Cosecant, and Cotangent Functions	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.12 Equivalent Representations of Trigonometric Functions	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
3.13 Trigonometry and Polar Coordinates	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
3.14 Polar Function Graphs	2–3	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.15 Rates of Change in Polar Functions	2–3	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 3.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	3.3	In pairs, students rehearse values of trigonometric functions at multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ using a modified question and answer-type game. Student *A* privately calculates $\sin\left(-\frac{\pi}{3}\right)$ and says, "The answer is $-\frac{\sqrt{3}}{2}$" and Student *B* asks a question such as "What is the sine of [some angle]?" or "What is the cosine of [some angle]? If Student *B* does not guess $\sin\left(-\frac{\pi}{3}\right)$, they work together to determine if Student *B*'s question is also valid for the given answer.
2	3.4	Students are given analytical trigonometric functions paired with graphs on a single sheet of paper or on index cards. Some of the graphs are correct, and others are incorrect. In pairs, students identify the function for which the analytical and graphical representations are consistent and those for which the representations are inconsistent. For graphs that are in error, students construct the appropriate graph.
3	3.6	In pairs, students are given a worksheet containing four functions such as $f(x) = 3\sin(x - \pi) - 4$ and $g(x) = -2\cos(3x) + 1$ along with a list of instructions such as: "Determine the midline," "Determine the period," "Determine the zeros on $[0,\ 2\pi]$," "On what intervals is the function increasing?" The size of the group is equal to the number of questions. Students take turns answering a question of their choice, passing the worksheet between turns. If the receiver of the worksheet disagrees with the passer's answer, they discuss it before the receiver answers a question and returns the worksheet. This continues until all questions are answered for all functions.
4	3.12	Provide each student with the same worksheet containing four trigonometric equations that require using equivalent trigonometric forms (such as sum identity or Pythagorean identity) to solve the equations, such as $2\sin^2 x + \cos x - 1 = 0$. Then, in an activity similar to the one used for Topic 3.6, arrange students in groups of four. Each student solves the first equation and then they pass their papers clockwise to the next student. Each student checks the first equation and, if necessary, discusses any mistakes with the previous student. Each student now solves the second equation on the paper, and the process continues until each student has their original paper back.

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Activity	Topic	Sample Activity
5	**3.14**	In groups of 2 or 3, give students a list of functions in rectangular form such as $f(x) = \sin(2x)$. Students graph the rectangular form on rectangular graph paper then use the information from this graph to sketch the corresponding polar function, $r = f(\theta) = \sin(2\theta)$ on polar graphing paper. Students discuss the relationships between the two graphs, describing intervals of increase and decrease, and intercepts.
6	**3.15**	Students are given a polar function. In pairs, students use their graphing calculator to determine intervals on which the function is positive and increasing, positive and decreasing, negative and increasing, and negative and decreasing. Students compare their answers and verify their results. Students then write, describing how the radius is changing as the angle is changing on each interval, and present this wording to the class.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 3.1

Periodic Phenomena

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.1.A Construct graphs of periodic relationships based on verbal representations.	**3.1.A.1** A periodic relationship can be identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals. **3.1.A.2** The graph of a periodic relationship can be constructed from the graph of a single cycle of the relationship.
3.1.B Describe key characteristics of a periodic function based on a verbal representation.	**3.1.B.1** The *period* of the function is the smallest positive value k such that $f(x+k)=f(x)$ for all x in the domain. Consequently, the behavior of a periodic function is completely determined by any interval of width k. **3.1.B.2** The period can be estimated by investigating successive equal-length output values and finding where the pattern begins to repeat. **3.1.B.3** Periodic functions take on characteristics of other functions, such as intervals of increase and decrease, different concavities, and various rates of change. However, with periodic functions, all characteristics found in one period of the function will be in every period of the function.

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TOPIC 3.2

Sine, Cosine, and Tangent

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

3.2.A

Determine the sine, cosine, and tangent of an angle using the unit circle.

ESSENTIAL KNOWLEDGE

3.2.A.1

In the coordinate plane, an angle is in *standard position* when the vertex coincides with the origin and one ray coincides with the positive x-axis. The other ray is called the *terminal ray*. Positive and negative angle measures indicate rotations from the positive x-axis in the counterclockwise and clockwise direction, respectively. Angles in standard position that share a terminal ray differ by an integer number of revolutions.

3.2.A.2

The radian measure of an angle in standard position is the ratio of the length of the arc of a circle centered at the origin subtended by the angle to the radius of that same circle. For a unit circle, which has radius 1, the radian measure is the same as the length of the subtended arc.

3.2.A.3

Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *sine* of the angle is the ratio of the vertical displacement of P from the x-axis to the distance between the origin and point P. Therefore, for a unit circle, the sine of the angle is the y-coordinate of point P.

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LEARNING OBJECTIVE

3.2.A

Determine the sine, cosine, and tangent of an angle using the unit circle.

ESSENTIAL KNOWLEDGE

3.2.A.4

Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *cosine* of the angle is the ratio of the horizontal displacement of P from the y-axis to the distance between the origin and point P. Therefore, for a unit circle, the cosine of the angle is the x-coordinate of point P.

3.2.A.5

Given an angle in standard position, the *tangent* of the angle is the slope, if it exists, of the terminal ray. Because the slope of the terminal ray is the ratio of the vertical displacement to the horizontal displacement over any interval, the tangent of the angle is the ratio of the y-coordinate to the x-coordinate of the point at which the terminal ray intersects the unit circle; alternately, it is the ratio of the angle's sine to its cosine.

TOPIC 3.3

Sine and Cosine Function Values

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

3.3.A

Determine coordinates of points on a circle centered at the origin.

ESSENTIAL KNOWLEDGE

3.3.A.1

Given an angle of measure θ in standard position and a circle with radius r centered at the origin, there is a point, P, where the terminal ray intersects the circle. The coordinates of point P are $\left(r\cos\theta, r\sin\theta\right)$.

3.3.A.2

The geometry of isosceles right and equilateral triangles, while attending to the signs of the values based on the quadrant of the angle, can be used to find exact values for the cosine and sine of angles that are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ radians and whose terminal rays do not lie on an axis.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 3.4

Sine and Cosine Function Graphs

Required Course Content

LEARNING OBJECTIVE

3.4.A
Construct representations of the sine and cosine functions using the unit circle.

ESSENTIAL KNOWLEDGE

3.4.A.1
Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine function, $f(\theta) = \sin\theta$, gives the y-coordinate, or vertical displacement from the x-axis, of point P. The domain of the sine function is all real numbers.

3.4.A.2
As the input values, or angle measures, of the sine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the vertical distance of points on the unit circle from the x-axis.

3.4.A.3
Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine function, $f(\theta) = \cos\theta$, gives the x-coordinate, or horizontal displacement from the y-axis, of point P. The domain of the cosine function is all real numbers.

3.4.A.4
As the input values, or angle measures, of the cosine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the horizontal distance of points on the unit circle from the y-axis.

TOPIC 3.5
Sinusoidal Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

3.5.A

Identify key characteristics of the sine and cosine functions.

ESSENTIAL KNOWLEDGE

3.5.A.1

A *sinusoidal function* is any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$. The sine and cosine functions are both sinusoidal functions, with $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.

3.5.A.2

The period and frequency of a sinusoidal function are reciprocals. The period of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 2π, and the frequency is $\frac{1}{2\pi}$.

3.5.A.3

The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. The amplitude of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 1.

3.5.A.4

The midline of the graph of a sinusoidal function is determined by the average, or arithmetic mean, of the maximum and minimum values of the function. The midline of the graphs of $y = \sin\theta$ and $y = \cos\theta$ is $y = 0$.

3.5.A.5

As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up.

3.5.A.6

The graph of $y = \sin\theta$ has rotational symmetry about the origin and is therefore an odd function. The graph of $y = \cos\theta$ has reflective symmetry over the y-axis and is therefore an even function.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 3.6

Sinusoidal Function Transformations

Required Course Content

LEARNING OBJECTIVE

3.6.A

Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.

ESSENTIAL KNOWLEDGE

3.6.A.1

Functions that can be written in the form $f(\theta) = a\sin(b(\theta + c)) + d$ or $g(\theta) = a\cos(b(\theta + c)) + d$, where a, b, c, and d are real numbers and $a \neq 0$, are sinusoidal functions and are transformations of the sine and cosine functions. Additive and multiplicative transformations are the same for both sine and cosine because the cosine function is a phase shift of the sine function by $-\frac{\pi}{2}$ units.

3.6.A.2

The graph of the additive transformation $g(\theta) = \sin\theta + d$ of the sine function $f(\theta) = \sin\theta$ is a vertical translation of the graph of f, including its midline, by d units. The same transformation of the cosine function yields the same result.

3.6.A.3

The graph of the additive transformation $g(\theta) = \sin(\theta + c)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units. The same transformation of the cosine function yields the same result.

3.6.A.4

The graph of the multiplicative transformation $g(\theta) = a\sin\theta$ of the sine function $f(\theta) = \sin\theta$ is a vertical dilation of the graph of f and differs in amplitude by a factor of $|a|$. The same transformation of the cosine function yields the same result.

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LEARNING OBJECTIVE

3.6.A

Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.

ESSENTIAL KNOWLEDGE

3.6.A.5

The graph of the multiplicative transformation $g(\theta) = \sin(b\theta)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. The same transformation of the cosine function yields the same result.

3.6.A.6

The graph of $y = f(\theta) = a\sin(b(\theta + c)) + d$ has an amplitude of $|a|$ units, a period of $\left|\frac{1}{b}\right|2\pi$ units, a midline vertical shift of d units from $y = 0$, and a phase shift of $-c$ units. The same transformations of the cosine function yield the same results.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 3.7

Sinusoidal Function Context and Data Modeling

Required Course Content

LEARNING OBJECTIVE

3.7.A

Construct sinusoidal function models of periodic phenomena.

ESSENTIAL KNOWLEDGE

3.7.A.1

The smallest interval of input values over which the maximum or minimum output values start to repeat, that is, the input-value interval between consecutive maxima or consecutive minima, can be used to determine or estimate the period and frequency for a sinusoidal function model.

3.7.A.2

The maximum and minimum output values can be used to determine or estimate the amplitude and vertical shift for a sinusoidal function model.

3.7.A.3

An actual pair of input-output values can be compared to pairs of input-output values produced by a sinusoidal function model to determine or estimate a phase shift for the model.

3.7.A.4.

Sinusoidal function models can be constructed for a data set with technology by estimating key values or using sinusoidal regressions.

3.7.A.5

Sinusoidal functions that model a data set are frequently only useful over their contextual domain and can be used to predict values of the dependent variable from values of the independent variable.

TOPIC 3.8

The Tangent Function

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.8.A Construct representations of the tangent function using the unit circle.	**3.8.A.1** Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The tangent function, $f(\theta) = \tan\theta$, gives the slope of the terminal ray. **3.8.A.2** Because the slope of the terminal ray is the ratio of the change in the y-values to the change in the x-values between any two points on the ray, the tangent function is also the ratio of the sine function to the cosine function. Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$.
3.8.B Describe key characteristics of the tangent function.	**3.8.B.1** Because the slope values of the terminal ray repeat every one-half revolution of the circle, the tangent function has a period of π. **3.8.B.2** The tangent function demonstrates periodic asymptotic behavior at input values $\theta = \frac{\pi}{2} + k\pi$, for integer values of k, because $\cos\theta = 0$ at those values. **3.8.B.3** The tangent function increases and its graph changes from concave down to concave up between consecutive asymptotes.

LEARNING OBJECTIVE

3.8.C

Describe additive and multiplicative transformations involving the tangent function.

ESSENTIAL KNOWLEDGE

3.8.C.1

The graph of the additive transformation $g(\theta) = \tan\theta + d$ of the tangent function $f(\theta) = \tan\theta$ is a vertical translation of the graph of f and the line containing its points of inflection by d units.

3.8.C.2

The graph of the additive transformation $g(\theta) = \tan(\theta + c)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units.

3.8.C.3

The graph of the multiplicative transformation $g(\theta) = a\tan\theta$ of the tangent function $f(\theta) = \tan\theta$ is a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

3.8.C.4

The graph of the multiplicative transformation $g(\theta) = \tan(b\theta)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

3.8.C.5

The graph of $y = f(\theta) = a\tan(b(\theta + c)) + d$ is a vertical dilation of the graph of $y = \tan\theta$ by a factor of $|a|$, has a period of $\left|\frac{1}{b}\right|\pi$ units, is a vertical shift of the line containing the points of inflection of the graph of $y = \tan\theta$ by d units, and is a phase shift of $-c$ units.

TOPIC 3.9

Inverse Trigonometric Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

3.9.A

Construct analytical and graphical representations of the inverse of the sine, cosine, and tangent functions over a restricted domain.

ESSENTIAL KNOWLEDGE

3.9.A.1

For inverse trigonometric functions, the input and output values are switched from their corresponding trigonometric functions, so the output value of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function.

3.9.A.2

The inverse trigonometric functions are called *arcsine*, *arccosine*, and *arctangent* (also represented as $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$). Because the corresponding trigonometric functions are periodic, they are only invertible if they have restricted domains.

3.9.A.3

In order to define their respective inverse functions, the domain of the sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the cosine function to $[0, \pi]$, and the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 3.10

Trigonometric Equations and Inequalities

Required Course Content

LEARNING OBJECTIVE

3.10.A
Solve equations and inequalities involving trigonometric functions.

ESSENTIAL KNOWLEDGE

3.10.A.1
Inverse trigonometric functions are useful in solving equations and inequalities involving trigonometric functions, but solutions may need to be modified due to domain restrictions.

3.10.A.2
Because trigonometric functions are periodic, there are often infinitely many solutions to trigonometric equations.

3.10.A.3
In trigonometric equations and inequalities arising from a contextual scenario, there is often a domain restriction that can be implied from the context, which limits the number of solutions.

TOPIC 3.11

The Secant, Cosecant, and Cotangent Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

3.11.A

Identify key characteristics of functions that involve quotients of the sine and cosine functions.

ESSENTIAL KNOWLEDGE

3.11.A.1

The secant function, $f(\theta) = \sec\theta$, is the reciprocal of the cosine function, where $\cos\theta \neq 0$.

3.11.A.2

The cosecant function, $f(\theta) = \csc\theta$, is the reciprocal of the sine function, where $\sin\theta \neq 0$.

3.11.A.3

The graphs of the secant and cosecant functions have vertical asymptotes where cosine and sine are zero, respectively, and have a range of $(-\infty, -1] \cup [1, \infty)$.

3.11.A.4

The cotangent function, $f(\theta) = \cot\theta$, is the reciprocal of the tangent function, where $\tan\theta \neq 0$. Equivalently, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$.

3.11.A.5

The graph of the cotangent function has vertical asymptotes for domain values where $\tan\theta = 0$ and is decreasing between consecutive asymptotes.

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 3.12

Equivalent Representations of Trigonometric Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.12.A Rewrite trigonometric expressions in equivalent forms with the Pythagorean identity.	**3.12.A.1** The Pythagorean Theorem can be applied to right triangles with points on the unit circle at coordinates $(\cos\theta, \sin\theta)$, resulting in the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$. **3.12.A.2** The Pythagorean identity can be algebraically manipulated into other forms involving trigonometric functions, such as $\tan^2\theta = \sec^2\theta - 1$, and can be used to establish other trigonometric relationships, such as $\arcsin x = \arccos\left(\sqrt{1 - x^2}\right)$, with appropriate domain restrictions.
3.12.B Rewrite trigonometric expressions in equivalent forms with sine and cosine sum identities.	**3.12.B.1** The sum identity for sine is $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. **3.12.B.2** The sum identity for cosine is $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. **3.12.B.3** The sum identities for sine and cosine can also be used as difference and double-angle identities. **3.12.B.4** Properties of trigonometric functions, known trigonometric identities, and other algebraic properties can be used to verify additional trigonometric identities.

continued on next page

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.12.C Solve equations using equivalent analytic representations of trigonometric functions.	**3.12.C.1** A specific equivalent form involving trigonometric expressions can make information more accessible. **3.12.C.2** Equivalent trigonometric forms may be useful in solving trigonometric equations and inequalities.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

TOPIC 3.13

Trigonometry and Polar Coordinates

Required Course Content

LEARNING OBJECTIVE

3.13.A
Determine the location of a point in the plane using both rectangular and polar coordinates.

ESSENTIAL KNOWLEDGE

3.13.A.1
The polar coordinate system is based on a grid of circles centered at the origin and on lines through the origin. Polar coordinates are defined as an ordered pair, (r, θ), such that $|r|$ represents the radius of the circle on which the point lies, and θ represents the measure of an angle in standard position whose terminal ray includes the point. In the polar coordinate system, the same point can be represented many ways.

3.13.A.2
The coordinates of a point in the polar coordinate system, (r, θ), can be converted to coordinates in the rectangular coordinate system, (x, y), using $x = r\cos\theta$ and $y = r\sin\theta$.

3.13.A.3
The coordinates of a point in the rectangular coordinate system, (x, y), can be converted to coordinates in the polar coordinate system, (r, θ), using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ for $x > 0$ or $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ for $x < 0$.

3.13.A.4
A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates. When the complex number has the rectangular coordinates $(a,\ b)$, it can be expressed as $a + bi$. When the complex number has polar coordinates $(r,\ \theta)$, it can be expressed as $(r\cos\theta) + i(r\sin\theta)$.

TOPIC 3.14

Polar Function Graphs

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.14.A Construct graphs of polar functions.	**3.14.A.1** The graph of the function $r = f(\theta)$ in polar coordinates consists of input-output pairs of values where the input values are angle measures and the output values are radii. **3.14.A.2** The domain of the polar function $r = f(\theta)$, given graphically, can be restricted to a desired portion of the function by selecting endpoints corresponding to the desired angle and radius. **3.14.A.3** When graphing polar functions in the form of $r = f(\theta)$, changes in input values correspond to changes in angle measure from the positive x-axis, and changes in output values correspond to changes in distance from the origin.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 3.15

Rates of Change in Polar Functions

Required Course Content

LEARNING OBJECTIVE

3.15.A

Describe characteristics of the graph of a polar function.

ESSENTIAL KNOWLEDGE

3.15.A.1

If a polar function, $r = f(\theta)$, is positive and increasing or negative and decreasing, then the distance between $f(\theta)$ and the origin is increasing.

3.15.A.2

If a polar function, $r = f(\theta)$, is positive and decreasing or negative and increasing, then the distance between $f(\theta)$ and the origin is decreasing.

3.15.A.3

For a polar function, $r = f(\theta)$, if the function changes from increasing to decreasing or decreasing to increasing on an interval, then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

3.15.A.4

The average rate of change of r with respect to θ over an interval of θ is the ratio of the change in the radius values to the change in θ over an interval of θ. Graphically, the average rate of change indicates the rate at which the radius is changing per radian.

3.15.A.5

The average rate of change of r with respect to θ over an interval of θ can be used to estimate values of the function within the interval.

AP PRECALCULUS

UNIT 4

Functions Involving Parameters, Vectors, and Matrices

Additional Topics Available to Schools
(not included on AP Precalculus Exam)

0%
AP EXAM WEIGHTING

35
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online optional **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 4 Part 1: Topics 4.1–4.7

Multiple-choice: 24
Free-response: 2

Progress Check Unit 4 Part 2: Topics 4.8–4.14

Multiple-choice: 21
Free-response: 2

UNIT 4

0% AP EXAM WEIGHTING

35 CLASS PERIODS

Functions Involving Parameters, Vectors, and Matrices

ESSENTIAL QUESTIONS

- How can we determine when the populations of species in an ecosystem will be relatively steady?
- How can we analyze the vertical and horizontal aspects of motion independently?
- How does high resolution computer-generated imaging achieve smooth and realistic motion on screen with so many pixels?

←→ Developing Understanding

In Unit 4, students explore function types that expand their understanding of the function concept. Parametric functions have multiple dependent variables' values paired with a single input variable or parameter. Modeling scenarios with parametric functions allows students to explore change in terms of components. This component-based understanding is important not only in calculus but in all fields of the natural and social sciences where we seek to understand one aspect of a phenomenon independent of other confounding aspects. Another major function type in this unit involves matrices mapping a set of input vectors to output vectors. The capacity to map large quantities of vectors instantaneously is the basis for vector-based computer graphics. While students may see their favorite video game character trip and fall or seemingly move closer or farther, matrices implement a rotation on a set of vectors or a dilation on a set of vectors. The power of matrices to map vectors is not limited to graphics but to any system that can be expressed in terms of components of vectors such as electrical systems, network connections, and regional population distribution changes over time. Vectors and matrices are also powerful tools of data science as they can be used to model aspects of complex scientific and social science phenomena.

Building the Mathematical Practices

2.A 2.B 3.A 3.C

When encountering new function types, students should engage with multiple representations of each function type and practice communicating precise characteristics of these function types. For parametric and vector-valued functions, students will need to use care in communicating about the position or velocity of an object, depending on the function that is given. Students should practice the precise language used with particle motion in the plane and refer specifically to position, direction, and motion. It will be valuable for students to provide clear rationales when setting up and working with matrices as linear transformation functions on vectors. Students should explain why they took the steps they did.

Preparing for the AP Exam

Unit 4 topics are excluded from the AP Exam. The AP Exam assesses topics in Unit 1, 2, and 3 as these topics are required for college credit and/or placement. When teachers and schools choose to include topics in Unit 4, students will use technology in ways that are new and unfamiliar. Students should practice setting up appropriate viewing windows and parameter restrictions when graphing parametric functions. Students should practice building matrices, manipulating matrices, and calculating the inverse of a matrix, where defined, with graphing calculators. Topic Questions and Progress Checks found in AP Classroom provide opportunities to practice with technology. Many of these practices will not only help students learn topics in this unit, but solidify understandings of topics in Units 1, 2, and 3.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
4.1 Parametric Functions	2	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
4.2 Parametric Functions Modeling Planar Motion	2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.B** Apply numerical results in a given mathematical or applied context.
4.3 Parametric Functions and Rates of Change	2	**3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
4.4 Parametrically Defined Circles and Lines	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.
4.5 Implicitly Defined Functions	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
4.6 Conic Sections	3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
4.7 Parametrization of Implicitly Defined Functions	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

continued on next page

UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
4.8 Vectors	3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.B** Apply numerical results in a given mathematical or applied context.
4.9 Vector-Valued Functions	1	**3.C** Support conclusions or choices with a logical rationale or appropriate data.
4.10 Matrices	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
4.11 The Inverse and Determinant of a Matrix	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
4.12 Linear Transformations and Matrices	1	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
4.13 Matrices as Functions	3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
4.14 Matrices Modeling Contexts	3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the optional **Progress Checks** for Unit 4.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	4.1	In pairs, students discuss why the graphs of parametric functions can fail the "vertical line test" (and yet they are functions), but the graphs of many other previously studied function types must pass the vertical line test.
2	4.2	Students are put into groups of 3-4 and are given a paper with a parametric function used to model planar motion. Each group gets a different function. Each person in the group writes a question on the paper that can be answered based on the given parametric function. On a separate blank sheet of paper, the group members verify that each question is answerable. Then, each group trades their question paper with another group. Each group works together to answer the questions on the received paper.
3	4.6	Students are provided with four graphs: a graph of a parabola that opens up or opens down with the vertex at the origin, a circle centered at the origin, an ellipse not centered at the origin, and a hyperbola with center at the origin. In groups, students develop a reason for why each of the four graphs does not belong to the set of four. (For example: The parabola is the only one that can be explicitly defined with y as a function of x; the circle is the only one with infinitely many lines of symmetry; the ellipse does not belong because it is not symmetric to the y-axis; the hyperbola does not belong because it has two disconnected pieces.) This type of activity can be repeated with four different graphs of a parabola, a circle, an ellipse, and a hyperbola, where students come up with the graphs.
4	4.8	Using information about the definition of the dot product, students construct an argument to explain why two nonzero vectors are perpendicular if, and only if, their dot product is equal to zero.
5	4.12	In pairs, students create a graphic organizer to highlight the similarities and differences between linear functions and linear transformations.
6	4.14	Students are given scenarios involving transitions between two states that can be modeled by matrices and linear transformations, such as the number of students who choose between two lunch options each day. For each model, in pairs, students explore past and future states for 2 steps using technology. Based on these limited data, each student tries to predict the steady state. Students then examine their hypotheses by calculating future states for 10, 100, and 1000 steps.

TOPIC 4.1
Parametric Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.A

Solve equations and inequalities represented analytically, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.1.A

Construct a graph or table of values for a parametric function represented analytically.

ESSENTIAL KNOWLEDGE

4.1.A.1

A parametric function in $\mathbb{R}^2$, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the *parameter*.

4.1.A.2

Because variables x and y are dependent on the independent variable, t, the coordinates (x_i, y_i) at time t_i can be written as functions of t and can be expressed as the single parametric function $f(t) = (x(t), y(t))$, where in this case x and y are names of two functions.

4.1.A.3

A numerical table of values can be generated for the parametric function $f(t) = (x(t), y(t))$ by evaluating x_i and y_i at several values of t_i within the domain.

4.1.A.4

A graph of a parametric function can be sketched by connecting several points from the numerical table of values in order of increasing value of t.

4.1.A.5

The domain of the parametric function f is often restricted, which results in start and end points on the graph of f.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.B

Apply numerical results in a given mathematical or applied context.

TOPIC 4.2

Parametric Functions Modeling Planar Motion

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.2.A

Identify key characteristics of a parametric planar motion function that are related to position.

ESSENTIAL KNOWLEDGE

4.2.A.1

A parametric function given by $f(t) = (x(t), y(t))$ can be used to model particle motion in the plane. The graph of this function indicates the position of a particle at time t.

4.2.A.2

The horizontal and vertical extrema of a particle's motion can be determined by identifying the maximum and minimum values of the functions $x(t)$ and $y(t)$, respectively.

4.2.A.3

The real zeros of the function $x(t)$ correspond to y-intercepts, and the real zeros of $y(t)$ correspond to x-intercepts.

TOPIC 4.3

Parametric Functions and Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.B

Apply numerical results in a given mathematical or applied context.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.3.A

Identify key characteristics of a parametric planar motion function that are related to direction and rate of change.

ESSENTIAL KNOWLEDGE

4.3.A.1

As the parameter increases, the direction of planar motion of a particle can be analyzed in terms of x and y independently. If $x(t)$ is increasing or decreasing, the direction of motion is to the right or left, respectively. If $y(t)$ is increasing or decreasing, the direction of motion is up or down, respectively.

4.3.A.2

At any given point in the plane, the direction of planar motion may be different for different values of t.

4.3.A.3

The same curve in the plane can be parametrized in different ways and can be traversed in different directions with different parametric functions.

4.3.A.4

Over a given interval $[t_1, t_2]$ within the domain, the average rate of change can be computed for $x(t)$ and $y(t)$ independently. The ratio of the average rate of change of y to the average rate of change of x gives the slope of the graph between the points on the curve corresponding to t_1 and t_2, so long as the average rate of change of $x(t) \neq 0$.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

TOPIC 4.4

Parametrically Defined Circles and Lines

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.4.A

Express motion around a circle or along a line segment parametrically.

ESSENTIAL KNOWLEDGE

4.4.A.1

A complete counterclockwise revolution around the unit circle that starts and ends at $(1,\ 0)$ and is centered at the origin can be modeled by $(x(t),\ y(t)) = (\cos t,\ \sin t)$ with domain $0 \le t \le 2\pi$.

4.4.A.2

Transformations of the parametric function $(x(t),\ y(t)) = (\cos t,\ \sin t)$ can model any circular path traversed in the plane.

4.4.A.3

A linear path along the line segment from the point $(x_1,\ y_1)$ to the point $(x_2,\ y_2)$ can be parametrized many ways, including using an initial position $(x_1,\ y_1)$ and rates of change for x with respect to t and y with respect to t.

TOPIC 4.5

Implicitly Defined Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.5.A Construct a graph of an equation involving two variables.	**4.5.A.1** An equation involving two variables can implicitly describe one or more functions. **4.5.A.2** An equation involving two variables can be graphed by finding solutions to the equation. **4.5.A.3** Solving for one of the variables in an equation involving two variables can define a function whose graph is part or all of the graph of the equation.
4.5.B Determine how the two quantities related in an implicitly defined function vary together.	**4.5.B.1** For ordered pairs on the graph of an implicitly defined function that are close together, if the ratio of the change in the two variables is positive, then the two variables simultaneously increase or both decrease; conversely, if the ratio is negative, then as one variable increases, the other decreases. **4.5.B.2** The rate of change of x with respect to y or of y with respect to x can be zero, indicating vertical or horizontal intervals, respectively.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 4.6

Conic Sections

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.6.A

Represent conic sections with horizontal or vertical symmetry analytically.

ESSENTIAL KNOWLEDGE

4.6.A.1

A parabola with vertex (h, k) can, if $a \neq 0$, be represented analytically as $x - h = a(y - k)^2$ if it opens left or right, or as $y - k = a(x - h)^2$ if it opens up or down.

4.6.A.2

An ellipse centered at (h, k) with horizontal radius a and vertical radius b can be represented analytically as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. A circle is a special case of an ellipse where $a = b$.

4.6.A.3

A hyperbola centered at (h, k) with vertical and horizonal lines of symmetry can be represented algebraically as $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for a hyperbola opening left and right, or as $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ for a hyperbola opening up and down. The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

TOPIC 4.7

Parametrization of Implicitly Defined Functions

Additional Topic Available to Schools

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.7.A Represent a curve in the plane parametrically.	**4.7.A.1** A parametrization $(x(t), y(t))$ for an implicitly defined function will, when $x(t)$ and $y(t)$ are substituted for x and y, respectively, satisfy the corresponding equation for every value of t in the domain. **4.7.A.2** If f is a function of x, then $y = f(x)$ can be parametrized as $(x(t), y(t)) = (t, f(t))$. If f is invertible, its inverse can be parametrized as $(x(t), y(t)) = (f(t), t)$ for an appropriate interval of t.
4.7.B Represent conic sections parametrically.	**4.7.B.1** A parabola can be parametrized in the same way that any equation that can be solved for x or y can be parametrized. Equations that can be solved for x can be parametrized as $(x(t), y(t)) = (f(t), t)$ by solving for x and replacing y with t. Equations that can be solved for y can be parametrized as $(x(t), y(t)) = (t, f(t))$ by solving for y and replacing x with t. **4.7.B.2** An ellipse can be parametrized using the trigonometric functions $x(t) = h + a\cos t$ and $y(t) = k + b\sin t$ for $0 \leq t \leq 2\pi$. **4.7.B.3** A hyperbola can be parametrized using trigonometric functions. For a hyperbola that opens left and right, the functions are $x(t) = h + a\sec t$ and $y(t) = k + b\tan t$ for $0 \leq t \leq 2\pi$. For a hyperbola that opens up and down, the functions are $x(t) = h + a\tan t$ and $y(t) = k + b\sec t$ for $0 \leq t \leq 2\pi$.

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INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.B

Apply numerical results in a given mathematical or applied context.

TOPIC 4.8

Vectors

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.8.A Identify characteristics of a vector.	**4.8.A.1** A vector is a directed line segment. When a vector is placed in the plane, the point at the beginning of the line segment is called the *tail*, and the point at the end of the line segment is called the *head*. The length of the line segment is the *magnitude* of the vector. **4.8.A.2** A vector $\overrightarrow{P_1P_2}$ with two components can be plotted in the xy-plane from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. The vector is identified by a and b, where $a = x_2 - x_1$ and $b = y_2 - y_1$. The vector can be expressed as $\langle a,\ b\rangle$. A zero vector $\langle 0,\ 0\rangle$ is the trivial case when $P_1 = P_2$. **4.8.A.3** The *direction* of the vector is parallel to the line segment from the origin to the point with coordinates $(a,\ b)$. The magnitude of the vector is the square root of the sum of the squares of the components. **4.8.A.4** For a vector represented geometrically in the plane, the components of the vector can be found using trigonometry.
4.8.B Determine sums and products involving vectors.	**4.8.B.1** The multiplication of a constant and a vector results in a new vector whose components are found by multiplying the constant by each of the components of the original vector. The new vector is parallel to the original vector.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.8.B Determine sums and products involving vectors.	**4.8.B.2** The sum of two vectors in $\mathbb{R}^2$ is a new vector whose components are found by adding the corresponding components of the original vectors. The new vector can be represented graphically as a vector whose tail corresponds to the tail of the first vector and whose head corresponds to the head of the second vector when the second vector's tail is located at the first vector's head. **4.8.B.3** The dot product of two vectors is the sum of the products of their corresponding components. That is, $\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle = a_1 a_2 + b_1 b_2$.
4.8.C Determine a unit vector for a given vector.	**4.8.C.1** A *unit vector* is a vector of magnitude 1. A unit vector in the same direction as a given nonzero vector can be found by scalar multiplying the vector by the reciprocal of its magnitude. **4.8.C.2** The vector $\langle a, b \rangle$ can be expressed as $a\vec{i} + b\vec{j}$ in $\mathbb{R}^2$, where $\vec{i}$ and $\vec{j}$ are unit vectors in the x and y directions, respectively. That is, $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.
4.8.D Determine angle measures between vectors and magnitudes of vectors involved in vector addition.	**4.8.D.1** The dot product is geometrically equivalent to the product of the magnitudes of the two vectors and the cosine of the angle between them. Therefore, if the dot product of two nonzero vectors is zero, then the vectors are perpendicular. **4.8.D.2** The Law of Sines and Law of Cosines can be used to determine side lengths and angle measures of triangles formed by vector addition.

INSTRUCTIONAL PERIODS: 1

SKILLS FOCUS

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 4.9

Vector-Valued Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.9.A

Represent planar motion in terms of vector-valued functions.

ESSENTIAL KNOWLEDGE

4.9.A.1

The position of a particle moving in a plane that is given by the parametric function $f(t) = (x(t), y(t))$ may be expressed as a *vector-valued function*, $p(t) = x(t)\vec{i} + y(t)\vec{j}$ or $p(t) = \langle x(t), y(t) \rangle$. The magnitude of the position vector at time t gives the distance of the particle from the origin.

4.9.A.2

The vector-valued function $v(t) = \langle x(t), y(t) \rangle$ can be used to express the velocity of a particle moving in a plane at different times, t. At time t, the sign of $x(t)$ indicates if the particle is moving right or left, and the sign of $y(t)$ indicates if the particle is moving up or down. The magnitude of the velocity vector at time t gives the speed of the particle.

TOPIC 4.10

Matrices

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B

Apply numerical results in a given mathematical or applied context.

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.10.A Determine the product of two matrices.	**4.10.A.1** An $n \times m$ matrix is an array consisting of n rows and m columns. **4.10.A.2** Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix. The product of the matrices is a new matrix in which the component in the ith row and jth column is the dot product of the ith row of the first matrix and the jth column of the second matrix.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 4.11

The Inverse and Determinant of a Matrix

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.11.A Determine the inverse of a 2×2 matrix.	**4.11.A.1** The identity matrix, I, is a square matrix consisting of 1s on the diagonal from the top left to bottom right and 0s everywhere else. **4.11.A.2** Multiplying a square matrix by its corresponding identity matrix results in the original square matrix. **4.11.A.3** The product of a square matrix and its inverse, when it exists, is the identity matrix of the same size. **4.11.A.4** The inverse of a 2×2 matrix, when it exists, can be calculated with or without technology.
4.11.B Apply the value of the determinant to invertibility and vectors.	**4.11.B.1** The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. The determinant can be calculated with or without technology and is denoted $\det(A)$. **4.11.B.2** If a 2×2 matrix consists of two column or row vectors from $\mathbb{R}^2$, then the nonzero absolute value of the determinant of the matrix is the area of the parallelogram spanned by the vectors represented in the columns or rows of the matrix. If the determinant equals 0, then the vectors are parallel. **4.11.B.3** The square matrix A has an inverse if and only if $\det(A) \neq 0$.

TOPIC 4.12

Linear Transformations and Matrices

INSTRUCTIONAL PERIODS: 1

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.12.A

Determine the output vectors of a linear transformation using a 2×2 matrix.

ESSENTIAL KNOWLEDGE

4.12.A.1

A *linear transformation* is a function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.

4.12.A.2

A linear transformation will map the zero vector to the zero vector.

4.12.A.3

A single vector in $\mathbb{R}^2$ can be expressed as a 2×1 matrix. A set of n vectors in $\mathbb{R}^2$ can be expressed as a $2 \times n$ matrix.

4.12.A.4

For a linear transformation, L, from $\mathbb{R}^2$ to $\mathbb{R}^2$, there is a unique 2×2 matrix, A, such that $L(\vec{v}) = A\vec{v}$ for vectors in $\mathbb{R}^2$. Conversely, for a given 2×2 matrix, A, the function $L(\vec{v}) = A\vec{v}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

4.12.A.5

Multiplication of a 2×2 transformation matrix, A, and a $2 \times n$ matrix of n input vectors gives a $2 \times n$ matrix of the n output vectors for the linear transformation $L(\vec{v}) = A\vec{v}$.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 4.13
Matrices as Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.13.A Determine the association between a linear transformation and a matrix.	**4.13.A.1** The linear transformation mapping $\langle x, y\rangle$ to $\langle a_{11}x + a_{12}y, a_{21}x + a_{22}y\rangle$ is associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. **4.13.A.2** The mapping of the unit vectors in a linear transformation provides valuable information for determining the associated matrix. **4.13.A.3** The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is associated with a linear transformation of vectors that rotates every vector an angle θ counterclockwise about the origin. **4.13.A.4** The absolute value of the determinant of a 2×2 transformation matrix gives the magnitude of the dilation of regions in $\mathbb{R}^2$ under the transformation.
4.13.B Determine the composition of two linear transformations.	**4.13.B.1** The composition of two linear transformations is a linear transformation. **4.13.B.2** The matrix associated with the composition of two linear transformations is the product of the matrices associated with each linear transformation.

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LEARNING OBJECTIVE

4.13.C

Determine the inverse of a linear transformation.

ESSENTIAL KNOWLEDGE

4.13.C.1

Two linear transformations are inverses if their composition maps any vector to itself.

4.13.C.2

If a linear transformation, L, is given by $L(\vec{v}) = A\vec{v}$, then its inverse transformation is given by $L^{-1}(\vec{v}) = A^{-1}\vec{v}$, where A^{-1} is the inverse of the matrix A.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 4.14

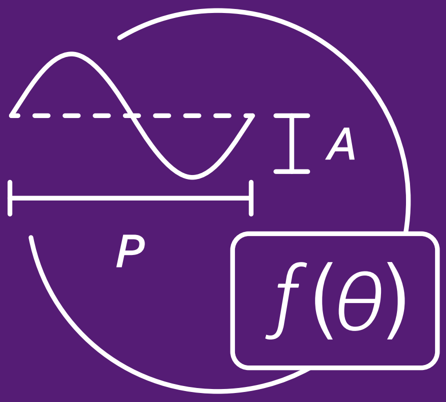
Matrices Modeling Contexts

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.14.A Construct a model of a scenario involving transitions between two states using matrices.	**4.14.A.1** A contextual scenario can indicate the rate of transitions between states as percent changes. A matrix can be constructed based on these rates to model how states change over discrete intervals.
4.14.B Apply matrix models to predict future and past states for n transition steps.	**4.14.B.1** The product of a matrix that models transitions between states and a corresponding state vector can predict future states. **4.14.B.2** Repeated multiplication of a matrix that models the transitions between states and corresponding resultant state vectors can predict the steady state, a distribution between states that does not change from one step to the next. **4.14.B.3** The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states.

AP PRECALCULUS

Instructional Approaches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Selecting and Using Course Materials

Textbooks

While the AP Program provides examples of textbooks to help determine whether a text is considered appropriate in meeting the AP Precalculus Course Audit curricular requirement, teachers select textbooks locally. **AP Central** has a list of textbook examples.

Graphing Calculators and Other Technologies in AP Precalculus

The use of a graphing calculator is considered an integral part of the AP Precalculus course, and it is required on some portions of the AP Exam. Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the National Academy of Sciences (NAS) Board on Mathematical Sciences and Their Applications have strongly endorsed the use of calculators in mathematics instruction and testing.

Graphing calculators are valuable tools for exploring multiple components of the mathematical practices, including using technology to develop function models that fit specified parameters or data, connecting concepts among their graphical, numerical, analytical, and verbal representations, and identifying and communicating accurate information. The AP Precalculus course also supports the use of other technologies that are available to students and encourages teachers to incorporate technology into instruction in a variety of ways as a means of facilitating discovery and reflection.

Appropriate examples of graphing calculator use in AP Precalculus include but certainly are not limited to:

- Graphing functions and analyzing their graphs to describe function behaviors (e.g., increasing/decreasing, concavity, maxima/minima, zeros, asymptotes).
- Generating a table of values for a function to develop an understanding of how the input-values and output-values change with respect to each other.
- Finding the real zeros of functions or points of intersection of the graphs of two functions to answer questions in context.
- Finding regressions equations to model data.
- Using matrix multiplication to explore the impact of a linear transformation on a vector (e.g., using matrices and inverse matrices to predict future and past states).

Accessible technology that has the capabilities expected for AP Precalculus is available for students who are blind or visually impaired. This technology should be used during the course, and an accommodation request to use this technology on the AP Exam must be made through the **College Board's Services for Students with Disabilities** (SSD).

Online Tools and Resources

In addition to the resources offered through your choice of textbook, many great and useful resources exist online—and are often free. Below are a few examples of these types of resources. This is not a comprehensive list, nor is it an endorsement of any of these resources. Sample different resources to find which ones can most benefit students' formative and summative assessments.

- **Classpad.net (Casio)**
- **Desmos**
- **GeoGebra**
- **Math Open Reference**
- **TI Education (Texas Instruments)**
- **WolframAlpha**
- **Wolfram MathWorld**

Professional Organizations

Professional organizations also serve as excellent resources for best practices and professional development opportunities. Following is a list of prominent organizations that serve the math education communities.

Mathematical Association of America
maa.org/
The MAA is the world's largest community of mathematicians, students, and enthusiasts. The organization's mission is to advance the understanding of mathematics and its impact on our world.

National Math and Science Initiative (NMSI)
nms.org
NMSI is a nonprofit organization whose mission is to improve student performance in the subjects of science, technology, engineering, and math (STEM) in the United States. They employ experienced AP teachers to train students and teachers in the STEM courses..

National Council of Teachers of Mathematics (NCTM)
nctm.org/
Founded in 1920, NCTM is the world's largest mathematics education organization, advocating for high-quality mathematics teaching and learning for all students.

Topic Notes

The following topic notes provide guidance on the following:

- The level of proficiency teachers should target after certain introductory topics
- Alternate ways to sequence instruction for certain topics
- Clarification on subtopics excluded from the framework

Unit 1

- **1.1 Change in Tandem:** Since this topic is introductory, students are not expected to become proficient in the ideas presented during the days devoted to this topic; they are revisited in later topics. Students are not expected to communicate formally within the days devoted to this topic but should grow in fluency as they progress from Topic Questions to Progress Checks to the AP Exam.
- **1.2 Rates of Change:** Since this topic is also introductory, students are not expected to become proficient in the ideas presented during the days devoted to this topic; they are revisited in later topics. Students are not expected to communicate formally within the days devoted to this topic but should grow in fluency as they progress from Topic Questions to Progress Checks to the AP Exam.
- **1.3 Rates of Change in Linear and Quadratic Functions:** Students should become proficient in the ideas presented during the days devoted to this topic. However, these topics will be revisited throughout the course.
- **1.4 Polynomial Functions and Rates of Change:** Although the formal definition of a polynomial function and the related terms should be part of the instruction of this topic, students should be able to recognize polynomial functions rather than provide a formal definition of them.
- **1.5 Polynomial Functions and Complex Zeros:** In this topic, factoring polynomials without technology is limited to factoring out a common factor and the rules for quadratics.
- **1.5 Polynomial Functions and Complex Zeros:** Special polynomial methods for finding roots, such as the rational root theorem and Descartes' rule of signs, are reserved for Algebra 2 and are not included in this topic.
- **1.6 Polynomial Functions and End Behavior:** Students are not expected to communicate formally within the days devoted to this topic but should grow in fluency as they progress from Topic Questions to Progress Checks to the AP Exam.
- **1.11 Equivalent Representations of Polynomial and Rational Expressions:** Partial fraction decomposition is not included in this course. Synthetic division is not required as part of polynomial long division.
- **1.12 Transformations of Functions:** The topic of transformations comes immediately before modeling, since, in some instances, students are expected to construct a model based on a "parent function" and transformations. However, this topic may occur anywhere in the unit prior to modeling.
- **1.12 Transformations of Functions:** Students will gradually mature into an understanding of one function being the transformation of another function—thinking in terms of two separate objects. However, their understanding and language at the beginning of study will likely be related to an "action view" of

transformation with the preimage being moved and stretched in the xy-plane. This is a useful tool for an early understanding of transformations. Students should use the geometric language of translations and dilations within this topic and beyond.

- **1.14 Function Model Construction and Application:** Students should be proficient in working with simple models within the days devoted to this topic. Increasingly complex models with other function types will be considered throughout the course.

Unit 2

- **2.1 Change in Arithmetic and Geometric Sequences:** Although series are formally outside of the scope of this framework, if state or district standards require instruction on series, this topic would be the appropriate place in the course to add the instruction.
- **2.6 Competing Function Model Validation:** In this topic, it is important for students to gain experience with using raw data sets and choosing an appropriate function model without being prompted toward a specific function type.
- **2.7 Composition of Functions:** Composition of functions and inverse functions could alternatively be addressed at the beginning of the unit, rather than in the middle. This approach might be preferred by some to avoid disrupting the flow of student experiences from exponential to logarithmic functions.
- **2.14 Logarithmic Function Context and Data Modeling:** The contextual interpretation of a and b in a logarithmic model $f(x) = a + b\log_n x$ is not included as part of the course.
- **2.15 Semi-log Plots:** For semi-log plots, logarithmic scaling will only be applied to the y-axis to linearize exponential functions, not the x-axis to linearize logarithmic functions.

Unit 3

- **3.4 Sine and Cosine Function Graphs-3.5 Sinusoidal Functions:** These topics might be intertwined instead of partitioned as the exploration of the graphs may induce language related to characteristics of the functions.
- **3.12 Equivalent Representations of Trigonometric Functions:** Half-angle identities, tangent sum/difference identities, and other identities not listed are not assessed on the AP Exam. However, Pythagorean identities, and sum, difference, and double-angle identities for sine and cosine *are* assessed on the AP Exam.

Unit 4: Not Assessed on the AP Exam

- **4.2 Parametric Functions Modeling Planar Motion:** The potential application of parametric functions is extensive but is limited to planar motion in this course.
- **4.6 Conic Sections:** The implicitly-defined functions study of conic sections does not include all of the characteristics found in a full analytical treatment of conic sections. It is limited to quantities found in the equations involving two variables that are needed to sketch graphs of the functions.
- **4.6 Conic Sections:** Conic sections are limited to those whose graphs have vertical and horizontal lines of symmetry.
- **4.8 Vectors:** The course framework limits matrices, vectors, and linear transformations to $\mathbb{R}^2$, but $\mathbb{R}^n$ can be explored if desired.
- **4.12 Linear Transformations and Matrices:** Discussion on the basis of a vector space is bypassed but can be developed if time allows since determining if vectors are linearly independent reinforces multiple topics such as determinants and inverses.

Developing the Mathematical Practices

Throughout the course, students will develop mathematical practices that are fundamental to the study of precalculus. Students will benefit from multiple opportunities to develop these practices in a scaffolded manner. The tables that follow look at each of the mathematical practices and their associated skills and provide examples of questions with sample activities for incorporating instruction on that skill into the course.

Mathematical Practice 1: Procedural and Symbolic Fluency: Algebraically manipulate functions, equations, and expressions

The table that follows provides examples of questions and sample activities for teaching students to successfully develop procedural and symbolic fluency for different topics throughout the course.

Mathematical Practice 1: *Procedural and Symbolic Fluency*

Skill	Questions to Ask Students	Sample Activities
1.A: *Solve equations and inequalities represented analytically, with and without technology.*	▪ What type of equation/inequality are we solving (linear, quadratic, radical, logarithmic, exponential, etc.)? ▪ Can technology help to solve it? ▪ What do the solutions represent? ▪ Are there any domain restrictions to consider? ▪ What strategy did you use to solve this equation/inequality? ▪ Does the solution make sense? How could the solution be verified? ▪ Can this equation be solved using a different method/strategy?	Give students a worksheet with a matching activity that has them match an inequality presented analytically, the solution set on a number line, and the solution set in interval notation. Include inequalities where the endpoints of solution intervals are both open, closed, or a combination. Have students sort a variety of cards containing equations and/or inequalities based on the type of function given in the problem. The type of function (i.e., radical, trigonometric, logarithmic, and exponential functions) will need different strategies to solve. Then have the students sort the cards into two groups—those that can be solved without technology vs. those for which the use of technology is best. Finally, have students pick one or two from each group and solve with or without technology, as appropriate. In small groups, have students identify what type of equation or inequality they are solving based on a running list of function types that students keep in their notebooks. Also have students individually try to solve the equation or inequality. Then, have students work in small groups to compare strategies and determine which are easier to use or preferred. For equations with many different paths toward a solution, split the entire class into teams, give each team a different solving technique, and conduct a mock debate over which method is better for solving. Have students work in groups of four where each student has an identical paper with the same four different equations on it. Have students complete the first one on their paper, and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Have students rotate again and the process continues until each student has their original paper back.

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Mathematical Practice 1: *Procedural and Symbolic Fluency (cont'd)*

Skill	Questions to Ask Students	Sample Activities
1.B: *Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.*	▪ How could analytical skills be used to change the form of the expression? ▪ Why would you want to change the function, equation, or expression into a different form? What would it help to find? ▪ What benefits come from alternate forms? ▪ How could you express this in a different form that would better convey the context? ▪ How do you know that these two forms are equivalent?	Give each student a paper with two columns that have exponential and logarithmic expressions in the first column and either an equivalent expression or one that is not in the second column. Have students state whether the two expressions are equivalent. This activity can also be done with trigonometric identities, rectangular and polar coordinates, etc. Develop a set of cards for several expressions. Some cards should have equivalent expressions to a starting expression, some should not. Divide students into small groups and give each group a set of cards. Groups must decide which cards display equivalent expressions and which do not belong. As an extension, have students determine how one form might be more useful than another. For polynomial and/or rational functions, have students plot the graph of the function by hand by generating coordinate pairs or by using graphing technology. Then have students factor the functions and compare characteristics of the graph to factored form. From this information, have students make conjectures about the relationship between the graph of a function and its factored form. Conclude this activity with instruction that formally connects the students' observations and the related mathematical properties of the function.

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Mathematical Practice 1: *Procedural and Symbolic Fluency (cont'd)*

Skill	Questions to Ask Students	Sample Activities
1.C: *Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.*	- What contextual clues/criteria/data are available to build a model? - How are the values involved changing with respect to each other? What type of equation best models this change? - What should the new function do or represent? What do you need to find or determine? - What parent functions can be used to construct a new function that helps in this situation? - What transformations could be applied to an existing model to model a new situation? - If technology is used for calculating regressions to build a model, what information does the technology need?	Randomly assign students into groups of 3. Each group is given the same modeling context and modeling function f, but groups have different assigned transformations of f, called g. Have students discuss the impact of the transformations $g(x) = f(x) + a$, $g(x) = f(x + a)$, $g(x) = af(x)$, or $g(x) = f(ax)$ on the context. Then, have students adjust the scenario to fit the new function g. As an extension, challenge students to find the inverse functions for both f and g, determine what the inverse functions model, and determine any necessary domain restrictions. Given a data set, have students work with a partner to construct an exponential function model of the form $f(x) = ab^x$. Students then present their data set and explain how they arrived at values for a and b. Have students use graphing technology to observe the changes and similarities when graphing a function and its additive and multiplicative transformations. Then have students generalize the impacts of the transformations to the graphs. Provide students with a function and have them find the inverse of their function using appropriate notation. Then have students use graphing technology to confirm that the two functions appear to be inverses. Give students the graph of an exponential or logarithmic function with transformations. Have students identify the parent function, write down the transformations to the parent function that they observe in the graph, and then switch with a partner and have their partner check their work. Working together, have the pair of students write each of their functions in analytical form. Finally, have students use graphing technology to confirm that the analytical form produces the correct graph.

Mathematical Practice 2: Multiple Representations: Translate mathematical information between representations

The table that follows provides examples of questions and sample activities for teaching students to understand and use multiple representations of functions for different topics throughout the course.

Mathematical Practice 2: *Multiple Representations*

Skill	Questions to Ask Students	Sample Activities
2.A: *Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.*	▪ What are we trying to answer? ▪ What information is given in the representation? ▪ What information in this function representation helps to identify the function type of the representation? ▪ What information do you need to know? Is there any information available that is not needed?	Give students a verbal description of a real-world scenario that can be modeled by a sinusoidal function. In groups, have students identify which aspects of the scenario relate to the amplitude, period, phase shift, and vertical shift of the related sinusoidal model. Have students examine a free-response type question that requires the use of technology to analyze a graph or table of values. First, have the students underline the questions being asked, and then have them write the analytical expression or equation that would be needed to answer each question. Have students switch papers with a partner so that their partner can check their work. As an extension, students work together to answer the question. Give students a representation of a function in a graph or table. Have each student write down a question that can be answered from the information in the graph or table. Then have students trade papers with a partner and answer the question asked. On the same piece of paper, have each student add another question that can be answered from the information in the graph or table. Then have students trade papers again, but with a different partner, to continue the process. Ensure that students know that each subsequent question needs to focus on a different aspect of the graph or table. Monitor student progress and allow the process to continue for as many times as needed to extract many key pieces of information from the representation. As a class, present students with different functions in various representations. Lead students in asking and answering questions that help identify the appropriate function type from the representation provided. Some examples include, but are not limited to: ▪ *Graphical representation*—What do you notice about the graph? Is there symmetry? Are the rates of change changing? Are there local min/max values? Intercepts? Asymptotes? What features do you notice? ▪ *Numerical representation*—Do we see any unusual values? Is there a pattern? Any symmetry? What are the output values doing as the input values increase? What does that mean for the function and what type(s) of function(s) is/are excluded with that feature? Are there local extrema? Asymptotic behavior? ▪ *Analytical representation*—What is our domain and range of the function? What other features are "easy" to see from this representation? ▪ *Verbal representation*—What key words/phrases do we see that might help us narrow down the type of model we need? Is there an indication about how quantities are changing with respect to each other? Are there any domain restrictions from this type of scenario?

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Mathematical Practice 2: *Multiple Representations (cont'd)*

Skill	Questions to Ask Students	Sample Activities
2.B: *Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.*	▪ Which representation would be the best in the situation? ▪ How can a different representation of a function be more useful? ▪ What would a graph of this situation look like? ▪ How could this graph or table be represented as an equation? ▪ How do you know these representations are equivalent?	Choose a function family and make sets of cards for functions within that family, where each set of cards has different representations of the functions: graphical, numerical, analytical, and verbal. Distribute the cards to the class and then have students find other students that have that same function in its different forms. Students should end up in small groups when they have all found their matches. Give students three of the four representations of a given function (i.e., graphical, numerical, analytical, and verbal). Have students work in pairs to construct the missing representation for each set. Have one pair of students collaborate with another pair to compare answers and make any final adjustments. Provide a Four Corners activity sheet with a trigonometric function in the center. If possible, each student should get a unique function. Ask students to express that trigonometric function graphically in one corner, numerically in another corner, analytically in another corner, and verbally in the last corner.

Mathematical Practice 3: Communication and Reasoning: Communicate with precise language, and provide rationales for conclusions

The table that follows provides examples of questions and example activities for helping students to develop the skills of communication and reasoning for different topics throughout the course. Well-communicated reasoning validates solutions.

Mathematical Practice 3: *Communication and Reasoning*

Skill	Questions to Ask Students	Sample Activities
3.A: *Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.*	▪ What does the function's graph look like? ▪ What are the zeros of the function? ▪ What are the domain and range of the function? ▪ What are the extrema of the function? ▪ Where is the function increasing or decreasing? ▪ How are the rates of change changing? ▪ What mathematical tools are available to help find different aspects of the function?	In groups of 3, have students describe characteristics of a function. They can include end behavior, zeros, intercepts, increasing/decreasing intervals, extrema, concavity, etc. Then, have each group present their characteristics of the function to the other groups for them to come up with a representation of the function. This can be done with any of the functions in the various units. Have students pair up and choose an "*A*" role and a "*B*" role. Let role *A* be the describer of certain characteristics they see from a graph they are given, but the role *B* student cannot see the graph. Role *B* must graph the function from *A*'s description without *A* seeing. Then let *A* see the result and then compare what could have been worded differently for better results. Then let *A* try again with a new function. Finally, students should switch roles and repeat the process. Have students work in small groups where each student is given a "secret" polynomial or rational function. Other students ask yes/no questions (e.g., Does your function's graph have a vertical asymptote?) to try to guess the "secret function."

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Mathematical Practice 3: *Communication and Reasoning (cont’d)*

Skill	Questions to Ask Students	Sample Activities
3.B: *Apply numerical results in a given mathematical or applied context.*	- Does the answer make sense in the context of the problem? - Is the answer in the domain of the function? - What does the result mean in the context of the problem? - Are there any numerical answers that would not make sense in the context of the problem? - What units are appropriate? - Is this solution reasonable? - What conclusions can we make from our results? Can we use them to make any predictions about other values?	Give students different scenarios with an appropriate function model and a solution to the function. In groups, have students discuss what the answer means in terms of the context and function. Give students a number of scenarios involving units. Instead of asking students to answer a question, have students work in pairs to write questions that would have different configurations of units as appropriate answers. For example, given a velocity function with units of meters per second, a student might write, “What is the average rate of change of velocity over a certain 3-second interval?” Present students with examples of logarithmic equations with work shown that leads to an erroneous result. Have students identify and correct the errors and identify the misconception that led to each error. Pairs of students are given tables of values within a context. The students calculate regression models with technology and make predictions about values within the given input-value range and beyond.
3.C: *Support conclusions or choices with a logical rationale or appropriate data.*	- What conclusion needs to be supported? - How can the conclusion be supported with a mathematical rationale? - Does the given data support your conclusion? - Can you convince others that your conclusion/choice makes sense? - What line of reasoning did you use to ...? - What evidence do you have to support ...? Is your evidence sufficient so support your conclusion? - What can you conclude from the evidence? - Is the data set appropriate as a model for the applied context? - Does the graph of the residuals appear without pattern?	Put students in groups of 3 and assign one group to be the “judges.” Then give all of the other groups a question that requires a reason to support the answer. Have students present their conclusion and supporting evidence to the judges to see which group has made the best argument. This activity can be used with any of the course topics. Give students several conclusions and a related data set. In pairs, students determine if each conclusion is supported by the data. Have students write concrete reasons for whether the data do or do not support the conclusion. Have students complete a fill-in-the-blank template that explains why not all pairs of matrices have products and the characteristics these matrices must possess to have a product. Blanks could include information about dimensions and how matrix multiplication works. Give the students multiple data sets. For each data set, have students calculate different regressions on the data set, graph the residuals, and decide if the function model is appropriate. Then, have students compare their answers to the results of a partner.

Instructional Strategies

The AP Precalculus course framework outlines the concepts and skills students must master in order to be successful on the AP Exam. In order to address those concepts and skills effectively, it helps to incorporate a variety of instructional approaches and best practices into daily lessons and activities. The following table presents strategies that can help students apply their understanding of course concepts.

Strategy	Definition	Examples
Ask the expert	Students are assigned as "experts" on concepts they understand well; groups rotate through the expert stations to learn about concepts they need to work on, providing students with opportunities to share knowledge and learn from one another.	When learning about regression models, assign students to become "experts" on the different types of regressions. Students rotate through stations in groups, working with the station expert to complete a series of problems using necessary steps with the technology and develop a greater understanding of that particular regression type.
Construct an Argument	Students use mathematical reasoning to present assumptions about mathematical situations, support conjectures with mathematically relevant and accurate data, and provide a logical progression of ideas leading to a conclusion that makes sense. This strategy can be used with word problems that do not lend themselves to immediate application of a formula or mathematical process.	Provide diameters of large oak trees and chainsaw blade speeds and ask students to construct a mathematical argument around how quickly a logger can cut up a giant oak tree, given a required set log length. Students can make an assumption about which logger will win a log cutting race if they have different chain saw speeds and diameters, and then find the solution and argue their case.
Create a Plan	Students analyze the tasks in a problem and create a process for completing the tasks. They find the information needed, interpret data, choose how to solve the problem, communicate the results, and verify accuracy.	Give students a variety of problems involving missing measurements in contextual settings with vectors that require the use of the Law of Sines or the Law of Cosines. Students must make sense of each problem, sketch the scenario utilizing vector addition, and represent unknowns. Students determine which law can be used in the scenario, and then calculate the missing measurement, making sense of their answer and interpreting the results. Students may write a general strategy for recognizing when each law should be applied and the steps to apply it.

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Strategy	Definition	Examples
Create Representations	Students create pictures, tables, graphs, lists, equations, models, and/or verbal expressions to interpret text or data.	In order to fully develop an understanding of composite functions, have students create a table for $f(g(x))$ by directly substituting output values for g as input values for f. Students should construct graphs of the functions and composition to see the relationships of the domains and ranges of the two functions. Students should also explore the algebraic representation of the composite function to compare to the graph and table.
Critique Reasoning	Through collaborative discussion, students respond to the arguments of others and question the use of mathematical terminology, assumptions, and conjectures to improve understanding and justify and communicate conclusions.	Given vector-valued functions, have students explain the direction of an object as it moves towards or away from another object with as accurate mathematical language as possible. As students discuss their responses in groups, they should help each other revise their communication of specific concepts using proper mathematical notation and terminology as well as justify their responses.
Debriefing	Students discuss their understanding of a concept to lead to a consensus on its meaning.	In order to determine if a function is invertible, have students discuss domain restrictions. Specifically, when working with trigonometric functions, students should discuss the domain and range of each function and address how to restrict the domain so that the function is invertible. This discussion will clarify misconceptions around domain and range and solidify the concept of functions and inverses.
Discussion Groups	Students work within groups to discuss content, create problem solutions, and explain and justify a solution.	Give groups of 3-4 students scenarios that can be modeled by linear, quadratic, polynomial, and piecewise-defined functions. Allow time for them to think independently and jot down aspects of the scenarios that would lend themselves to different types of models (i.e., how one quantity changes in relation to another) and then share their reasoning with the group behind their model selection and discussing potential domain and range restrictions based on contextual clues.
Distractor Analysis	Students examine answers to a multiple-choice question and determine which answers are incorrect and why.	To help students understand the graphs of rational functions' vertical asymptotes and holes, provide multiple-choice questions with a set of "distractor answers," in addition to the correct answer, along with a separate list of rationales—one for each answer choice—without telling students which rationale applies to which answers. For example, if the question asks for the vertical asymptote of the graph of $f(x) = \dfrac{x^2 - 1}{x^2 - 5x - 6}$, then one distractor answer could be $x = -1$ and its corresponding rationale could be, "This choice is for the student who factored correctly but found the hole, rather than the vertical asymptote." Have students match each distractor answer choice to its corresponding rationale in the list.

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Strategy	Definition	Examples
Error Analysis	Students analyze an existing solution to determine whether (or where) errors have occurred.	When students are working with parametrization of implicitly-defined functions, give pairs of students the steps toward and a final equation after eliminating the parameter. Each "solution" presented contains an error in the process. Students individually identify the error then confirm with a partner. After confirming the error, they complete the correct process individually and compare their results.
Four Corners	Students are given a sheet of paper that's been divided vertically and horizontally with four equal sections and a topic, concept, or function in the center of the page. Each section is used to provide information about the center item (e.g., different representations of a function).	To help students learn about each conic section, provide a Four Corners activity sheet for each conic and have students determine the features of the conic in one corner, graph in another corner, a table in another corner, and a verbal description of the features in the last corner.
Graph and Switch	Each student in a pair generates a graph (or sketch of a graph) to model a certain function. Then, the students switch graphing calculators (or papers) to review each other's solutions.	As students learn about sinusoidal functions, have them graph a sinusoidal function of their choice. Then, they can switch graphing calculators with a partner and see if they can generate the analytic representation of the graph their partner created.
Graphic Organizer	Students represent ideas and information visually (e.g., Venn diagrams, flowcharts, etc.).	In order to determine the set of solutions to rational inequalities, have students construct a table defining appropriate test values in intervals defined by boundary points. Then, students will mark the correct intervals and incorrect intervals.
Identify a Subtask	Students break a problem into smaller pieces whose combined outcomes lead to a solution.	As students learn to build trigonometric models from scenarios, have them break the task into finding the period, amplitude, vertical shift, and phase shift.
Look for a Pattern	Students observe information or create visual representations to find a trend.	To show that patterns can be detected when graphing a cycle of sine or cosine, have students calculate the exact values of multiples of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, in order to determine the correct graph within the bounds.

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Strategy	Definition	Examples
Odd One Out	In groups of four, students are given four problems, images, or graphs. Three of the items should have something in common. Each student in the group works with one of the four items and must decide individually whether their item fits with the other three items and then write a reason why. Students then share their responses within their groups.	Begin by modeling an example, such as three images of four-legged animals and one image of an object, explaining why the object is the "odd one out." Divide students into groups of four and then provide each student in the group with one of four images displaying graphs. Three of the graphs should represent exponential functions, while the fourth graph represents a logarithmic function. Students examine their graph and write on mini whiteboards or sheets of paper, "I'm in because …" or "I'm out because …." All members of each group discuss their answers together and signal when they have reached consensus. Then reveal the answer and explain what the three similar graphs had in common, and why the other was the odd one out.
Predict and Confirm	Students make conjectures and confirm or modify the conjectures based on outcomes of an activity.	Give each student two sets of cards: one with the trigonometric functions (sine, cosine, and tangent) and the other with the graphs of the reciprocal functions (cosecant, secant, and cotangent). Then, have students attempt to match the function with their appropriate reciprocal graph. Students should find the relationship between roots of the functions and asymptotes of the reciprocal functions' graphs. They can also compare the range of the function to the range of the graph. Finally, students can use graphing calculators to confirm that they have matched the correct function to each reciprocal function graph.
Quickwrite	Students write for a short, specific amount of time about a designated topic.	To help synthesize concepts after learning how to work with rational functions, have students write as much as they can about the similarities and differences between rational functions with holes and vertical asymptotes in 3–5 minutes. As part of a share-out, have students summarize how to tell if the graph of the function has a vertical asymptote or a hole.
Quiz-Quiz-Trade	Students answer a question independently, and then quiz a partner on the same question.	Give each student a card containing a question on the front and have them solve the problem, showing their work and answer, on the back. Each student should get a different question. At regular time intervals, have students move around the room to partner with a classmate. Then, the students in each pair take turns quizzing each other about their individual problems. When you signal that time is up, students switch cards with each other—so they now have a different question than the one they started with—and find a new partner. Have the process repeat for a number of rounds you determine.

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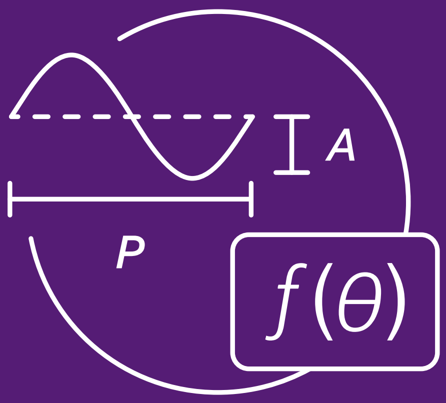
Strategy	Definition	Examples
Round Table	Students are given a worksheet with multiple problems, or a single problem with many parts and work individually to solve one assigned to them. Then, students rotate after each problem, or part of the single problem, to allow other group members to check their work.	In groups of four, have each student use a different color pen/pencil to work a problem on a worksheet one step at a time. Student *A* works part one of the problem and then passes the paper to Student *B*. In turn, Student *B* checks the work of Student *A*, making corrections if needed, and then completes part two of the problem. Have students rotate again and continue the process until the problem is solved. Each group in the class could work the same problem, or for variation, each group could have a different problem. After a set amount of time, the teacher can call for one student from each group to rotate to a new group and share their original problem with the new group members. This activity could be used for a variety of solving equations topics.
Simplify the Problem	Students use "friendlier" numbers or functions to help solve a problem.	When discussing equivalent trigonometric functions, review equivalent trigonometric models using "friendly numbers" before moving on to problems where the transformations are more complex. For example, use multiples of $\frac{\pi}{6}$ for exploring the sine and cosine sum identities before other angles.
Think Aloud	Students talk through a difficult problem by describing what the text means.	In order to determine which transformations occurred to a given function, have students ask themselves a series of questions out loud to identify each type of transformation. These questions could include but are not limited to: Has the graph of the function been vertically stretched? Has the graph been horizontally or vertically shifted? You may want to provide students with a list of questions to start and then have them add their own question(s).
Think-Pair-Share (or Wait, Turn, and Talk)	Students think through a problem alone, pair with a partner to share ideas, and then conclude by sharing results with the class.	Given the equations of several exponential functions, have students think of the steps needed to solve for a value located in the exponent. Then, have students share their ideas with a partner before sharing out to the whole class.

AP PRECALCULUS

Exam Information


A
P
f(θ)

Exam Overview

The AP Precalculus Exam assesses student understanding of the mathematical practices and learning objectives outlined in the course framework inclusive of Units 1, 2, and 3. The exam is 3 hours long and includes 40 multiple-choice questions and 4 six-point free-response questions, each weighted equally and scored on an analytic scale. The details of the exam, including exam weighting, timing, and calculator requirements, can be found below:

Section	Question Type	Number of Questions	Exam Weighting	Timing
I	**Multiple-choice questions**			
	Part A: No calculator is permitted	28	43.75%	80 minutes
	Part B: Graphing calculator required	12	18.75%	40 minutes
II	**Free-response questions**			
	Part A: Graphing calculator required	2	18.75%	30 minutes
	Part B: No calculator is permitted	2	18.75%	30 minutes

Exam Weighting for Skills

	Exam Weighting
Practice 1: Procedural and Symbolic Fluency	
1.A: Solve equations and inequalities	**14–17%**
1.B: Express equivalent forms	**9–13%**
1.C: Construct new functions	**15–19%**
Practice 2: Multiple Representations	
2.A: Identify information from representations	**14–17%**
2.B: Construct equivalent representations	**6–9%**
Practice 3: Communication and Reasoning	
3.A: Describe characteristics	**10–14%**
3.B: Apply results	**9–13%**
3.C: Support conclusions	**13%**

How Student Learning Is Assessed on the AP Exam

Section I: Multiple-Choice

The first section of the AP Precalculus Exam includes 40 multiple-choice questions. A graphing calculator is required for the final 12 questions (Part B). Calculators should be in radian mode for the AP Exam. Only some of the Part B questions will require the use of the graphing calculator. Refer to the "Technology Needs" section on page 7 for a list of calculator capabilities that are necessary for the AP Exam. The multiple-choice section includes items from different representations (graphical, numerical, analytical, and verbal), items in a real-world context, and items involving modeling.

In addition, the various function types found in the AP Precalculus framework are assessed based on the following weights in the multiple-choice section.

Function Type	Unit	MCQ Section Weighting
General Functions (non-analytical)	**Units 1 and 2**	**15–23%**
Polynomial and Rational Functions	**Unit 1**	**20–25%**
Exponential and Logarithmic Functions	**Unit 2**	**22–28%**
Trigonometric and Polar Functions	**Unit 3**	**30–35%**

Section II: Free-Response

The 4 six-point free-response questions on the AP Precalculus Exam include content topics from Units 1, 2, and 3. Two of the four questions incorporate a real-world function modeling context. A graphing calculator is required for the first 2 questions (Part A). Calculators should be in radian mode for the AP Exam. Refer to the "Technology Needs" section on page 7 for a list of calculator capabilities that are necessary for the AP Exam.

Free-Response Task Model Overview

Free-Response Task Type	Unit Focus	Graphing Calculator?	Real-World Context?
FRQ 1: Function Concepts	1, 2	Yes	No
FRQ 2: Modeling a Non-Periodic Context	1, 2	Yes	Yes
FRQ 3: Modeling a Periodic Context	3	No	Yes
FRQ 4: Symbolic Manipulations	2, 3	No	No

Section II: Free-Response Point Count by Skill

	1.A	1.B	1.C	2.A	2.B	3.A	3.B	3.C
FRQ 1	1	0	1	2	0	1	0	1
FRQ 2	0	1	2	0	0	0	1	2
FRQ 3	0	0	2	1	2	1	0	0
FRQ 4	4	2	0	0	0	0	0	0

About the Free-Response Questions

A walk-through video is available for each free-response question that follows in this AP Course and Exam Description. Each video provides teachers with additional information that will help to inform their instruction. These videos can be accessed **here**.

FRQ 1: Function Concepts (Graphing Calculator) presents functions expressed graphically, numerically, and analytically. The question includes three Parts and requires students to work with a variety of concepts. These may include function composition, inverse functions, function input-output values, zeros of a function, end behavior of a function, and identification of an appropriate function type to construct a function model. The graphing calculator is useful in this question. An instructional video on FRQ 1 can be found **here**.

FRQ 2: Modeling a Non-Periodic Context (Graphing Calculator) presents a real-life context. In Part (A), students use the given information to construct a function model by building a system of equations and finding the parameters using a method of choice. Function types include polynomial, piecewise-defined, exponential, and logarithmic. In Part (B), students calculate, apply, and reason with average rates of change and their units. In Part (C), students justify a conclusion about assumptions or limitations of the model. The graphing calculator is useful in this question. An instructional video on FRQ 2 can be found **here**.

FRQ 3: Modeling a Periodic Context (No Calculator) presents a real-life context that is modeled by a sinusoidal function. In Part (A), students use the given information to identify coordinates of five labeled points on the graph of the sinusoidal function and its midline for two full cycles. In Part (B), students find the parameters of an analytical presentation of the sinusoidal function. Both Parts (A) and (B) require students to construct the sinusoidal model by using the context to determine the vertical dilation and vertical translation of the sine or cosine function (amplitude and vertical shift), and the horizontal dilation and horizontal translation of the sine or cosine function (period and phase shift). In Part (C), students answer questions about the behavior of the function, the concavity of the graph, and whether the rate of change is increasing or decreasing on a particular interval. An instructional video on FRQ 3 can be found **here**.

FRQ 4: Symbolic Manipulations (No Calculator) presents several functions: exponential, logarithmic, trigonometric, and/or inverse trigonometric. Two parts of the question require students to solve equations using given functions. The third part of the question requires students to rewrite given function expressions in equivalent forms. In this question, students must (1) determine the exact values of expressions that can be obtained without a calculator (2) use algebraic methods and rules for exponents and logarithms to combine terms (3) show the work that leads to their answers in each part of the question. An instructional video on FRQ 4 can be found **here**.

General Instructions for the Free-Response Questions

The following instructions will be provided to students for their AP Exam. Teachers are encouraged to share and discuss these with students so that students understand the requirements for the AP Exam.

- Show all of your work. Your work will be scored on the correctness and completeness of your responses, including your supporting work and answers. Answers without supporting work may not receive credit in cases where supporting work is requested.
- During Part A, you are expected to use your graphing calculator for tasks such as producing graphs and tables, evaluating functions, solving equations, and performing computations.
- **For Part A, your calculator must be in radian mode.** Avoid rounding intermediate computations on the way to the final result. Unless otherwise specified, any decimal approximations reported in your work should be accurate to three places after the decimal point.
- For Part A, it may be helpful to use your graphing calculator to store information such as computed values for constants, functions you are working with, solutions to equations, and any intermediate values. Computations with the graphing calculator that use the stored information help to maintain as much precision as possible and ensure the desired accuracy in final answers.
- During Part B, no calculator is allowed. Carefully read the instructions provided with the questions.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Task Verbs Used in Free-Response Questions

The following task verbs are commonly used in the free-response questions:

- **Construct/write a function/expression/equation/model:** Develop an analytical representation, with and without technology, that is consistent with a scenario, data set, or other criteria.
- **Describe:** Develop a verbal representation that is consistent with a scenario, data set, function representation, or other criteria.
- **Determine/Find/Identify:** Apply appropriate methods or processes for answering a question.
- **Estimate/Compare:** Use a function representation to find approximate values and/or compare results.
- **Explain/Give a reason/Provide a rationale/Justify:** Use information from the scenario or function representation to provide reasons or rationales for solutions or conclusions.
- **Express/Indicate:** Provide information or a result in a desired form or include units as part of the answer to a question.
- **Interpret:** Describe the connection between a mathematical expression or solution and its meaning within the realistic context of a problem, often including consideration of units.
- **Plot and label, sketch and label:** Develop a graphical representation that is consistent with a scenario, data set, or other criteria.
- **Rewrite:** Apply appropriate methods to determine/find equivalent analytical representations of an expression.
- **Solve:** Apply appropriate methods to determine/find solutions to an equation or inequality.

Sample AP Precalculus Exam Questions

The sample exam questions that follow illustrate the relationship between the course framework and the AP Precalculus Exam and serve as examples of the types of questions that appear on the exams. After the sample questions is a table which shows which skill, learning objective(s), and unit each question relates to. The table also provides the answers to the multiple-choice questions.

Section I: Multiple-Choice

PART A

No calculator is allowed for this part of the exam.

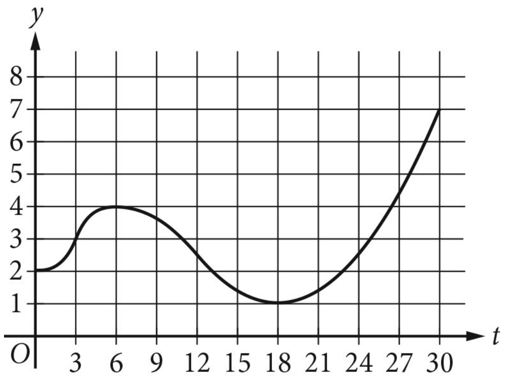
1. The polynomial function p is given by $p(x) = -4x^5 + 3x^2 + 1$. Which of the following statements about the end behavior of p is true?

   (A) The sign of the leading term of p is positive, and the degree of the leading term of p is even; therefore, $\lim_{x \to -\infty} p(x) = \infty$ and $\lim_{x \to \infty} p(x) = \infty$.

   (B) The sign of the leading term of p is negative, and the degree of the leading term of p is odd; therefore, $\lim_{x \to -\infty} p(x) = \infty$ and $\lim_{x \to \infty} p(x) = -\infty$.

   (C) The sign of the leading term of p is positive, and the degree of the leading term of p is odd; therefore, $\lim_{x \to -\infty} p(x) = -\infty$ and $\lim_{x \to \infty} p(x) = \infty$.

   (D) The sign of the leading term of p is negative, and the degree of the leading term of p is odd; therefore, $\lim_{x \to -\infty} p(x) = -\infty$ and $\lim_{x \to \infty} p(x) = \infty$.



Graph of W

2. The depth of water, in feet, at a certain place in a lake is modeled by a function W. The graph of $y = W(t)$ is shown for $0 \leq t \leq 30$, where t is the number of days since the first day of a month. What are all intervals of t on which the depth of water is increasing at a decreasing rate?

   (A) $(3, 6)$ only

   (B) $(3, 12)$

   (C) $(0, 3)$ and $(18, 30)$ only

   (D) $(0, 6)$ and $(18, 30)$

3. Which of the following functions has a zero at $x = 3$ and has a graph in the xy-plane with a vertical asymptote at $x = 2$ and a hole at $x = 1$?

   (A) $h(x) = \dfrac{x^2 - 4x + 3}{x^2 - 3x + 2}$

   (B) $j(x) = \dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$

   (C) $k(x) = \dfrac{x - 3}{x^2 - 3x + 2}$

   (D) $m(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

4. The polynomial function p is an odd function. If $p(3) = -4$ is a relative maximum of p, which of the following statements about $p(-3)$ must be true?

   (A) $p(-3) = 4$ is a relative maximum.

   (B) $p(-3) = -4$ is a relative maximum.

   (C) $p(-3) = 4$ is a relative minimum.

   (D) $p(-3) = -4$ is a relative minimum.

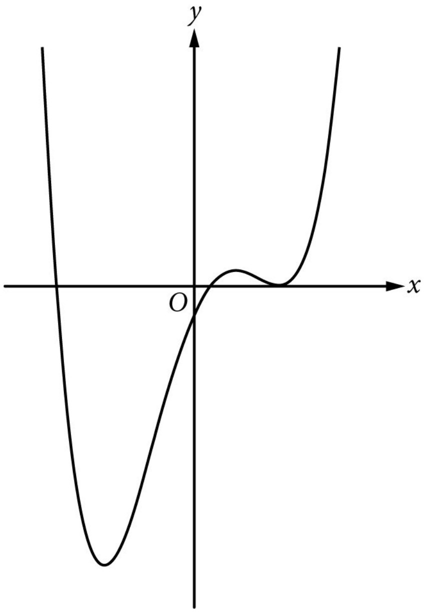
5. The function g is given by $g(x) = x^3 - 3x^2 - 18x$, and the function h is given by $h(x) = x^2 - 2x - 35$. Let k be the function given by $k(x) = \frac{h(x)}{g(x)}$. What is the domain of k ?

(A) all real numbers x where $x \neq 0$

(B) all real numbers x where $x \neq -5,\ x \neq 7$

(C) all real numbers x where $x \neq -3,\ x \neq 0,\ x \neq 6$

(D) all real numbers x where $x \neq -5,\ x \neq -3,\ x \neq 0,\ x \neq 6,\ x \neq 7$



Graph of g

6. The figure shown is the graph of a polynomial function g. Which of the following could be an expression for $g(x)$?

(A) $0.25(x-5)(x-1)(x+8)$

(B) $0.25(x+5)(x+1)(x-8)$

(C) $0.25(x-5)^2(x-1)(x+8)$

(D) $0.25(x+5)^2(x+1)(x-8)$

x	-8	-4	-2	-1	0	3
$f(x)$	87	55	5	-4	-7	20

7. The table gives values for a polynomial function f at selected values of x. Let $g(x) = af(bx) + c$, where a, b, and c are positive constants. In the xy-plane, the graph of g is constructed by applying three transformations to the graph of f in this order: a horizontal dilation by a factor of 2, a vertical dilation by a factor of 3, and a vertical translation by 5 units. What is the value of $g(-4)$?

   (A) 266

   (B) 170

   (C) 28

   (D) 20

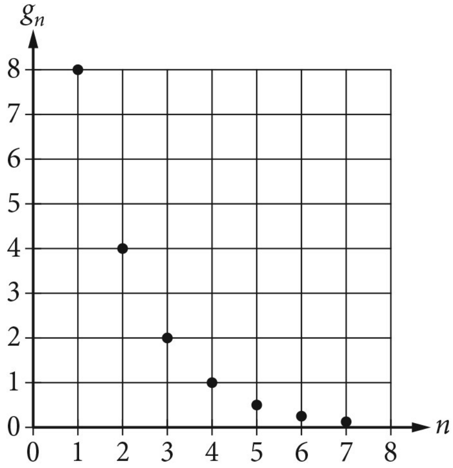
8. Let k, w, and z be positive constants. Which of the following is equivalent to $\log_{10}\left(\frac{kz}{w^2}\right)$?

   (A) $\log_{10}(k+z) - \log_{10}(2w)$

   (B) $\log_{10} k + \log_{10} z - 2\log_{10} w$

   (C) $\log_{10} k + \log_{10} z - \frac{1}{2}\log_{10} w$

   (D) $\log_{10} k - \log_{10} z + 2\log_{10} w$



9. Values of the terms of a geometric sequence g_n are graphed in the figure. Which of the following is an expression for the nth term of the geometric sequence?

   (A) $g_n = 4\left(\frac{1}{2}\right)^{(n-2)}$

   (B) $g_n = 8(2)^{(n-1)}$

   (C) $g_n = 8\left(\frac{1}{2}\right)^{n}$

   (D) $g_n = 16\left(\frac{1}{2}\right)^{(n-1)}$

x	$g(x)$
−2	4
0	$\frac{1}{2}$
3	−2
4	3
36	9

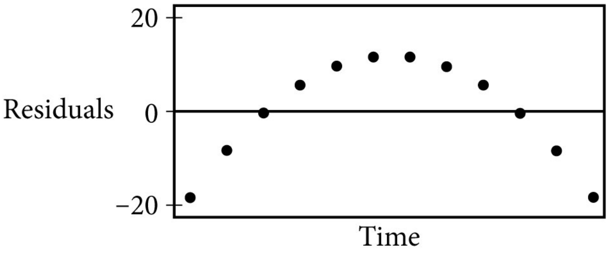
10. The table gives values of the function g for selected values of x. The function f is given by $f(x) = 3^x + x^2$. What is the value of $f(g(3))$?

(A) −72

(B) $\frac{37}{9}$

(C) 9

(D) 97



11. A food vendor developed a new sandwich type for sale. The vendor made estimates about the sales of the new sandwich type over time. A linear regression was used to develop a model for the sales over time. The figure shows a graph of the residuals of the linear regression. Which of the following statements about the linear regression is true?

(A) The linear model is not appropriate, because there is a clear pattern in the graph of the residuals.

(B) The linear model is not appropriate, because the graph of the residuals has more points above 0 than below 0.

(C) The linear model is appropriate, because there is a clear pattern in the graph of the residuals.

(D) The linear model is appropriate, because the positive residual farthest from 0 and the negative residual farthest from 0 are about the same distance, although more points are above 0 than below 0.

12. The value, in millions of dollars, of transactions processed by an online payment platform is modeled by the function M. The value is expected to increase by 6.1% each quarter of a year. At time $t = 0$ years, 54 million dollars of transactions were processed. If t is measured in years, which of the following is an expression for $M(t)$? (Note: A quarter is one fourth of a year.)

(A) $54(0.061)^{(t/4)}$

(B) $54(0.061)^{(4t)}$

(C) $54(1.061)^{(t/4)}$

(D) $54(1.061)^{(4t)}$

13. Iodine-131 has a half-life of 8 days. In a particular sample, the amount of iodine-131 remaining after d days can be modeled by the function h given by $h(d) = A_0(0.5)^{(d/8)}$, where A_0 is the amount of iodine-131 in the sample at time $d = 0$. Which of the following functions k models the amount of iodine-131 remaining after t hours, where A_0 is the amount of iodine-131 in the sample at time $t = 0$? (There are 24 hours in a day, so $t = 24d$.)

(A) $k(t) = A_0(0.5)^{(t/24)}$

(B) $k(t) = A_0\left(0.5^{(1/24)}\right)^{(8t)}$

(C) $k(t) = A_0\left(0.5^{(24)}\right)^{(t/8)}$

(D) $k(t) = A_0\left(0.5^{(1/192)}\right)^{t}$

14. What are all values of x for which $\ln\left(x^3\right) - \ln x = 4$?

(A) $x = -2$ and $x = 2$

(B) $x = -e^2$ and $x = e^2$

(C) $x = e^2$ only

(D) $x = e^4$

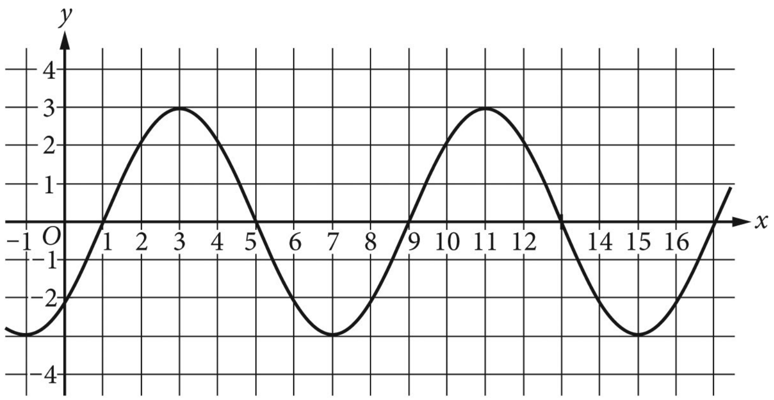
15. Let $f(x) = 1 + 3\sec x$ and $g(x) = -5$. In the xy-plane, what are the x-coordinates of the points of intersection of the graphs of f and g for $0 \le x < 2\pi$?

(A) $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

(B) $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

(C) $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$

(D) $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$



Graph of g

16. The figure shows the graph of a sinusoidal function g. What are the values of the period and amplitude of g?

(A) The period is 4, and the amplitude is 3.

(B) The period is 8, and the amplitude is 3.

(C) The period is 4, and the amplitude is 6.

(D) The period is 8, and the amplitude is 6.

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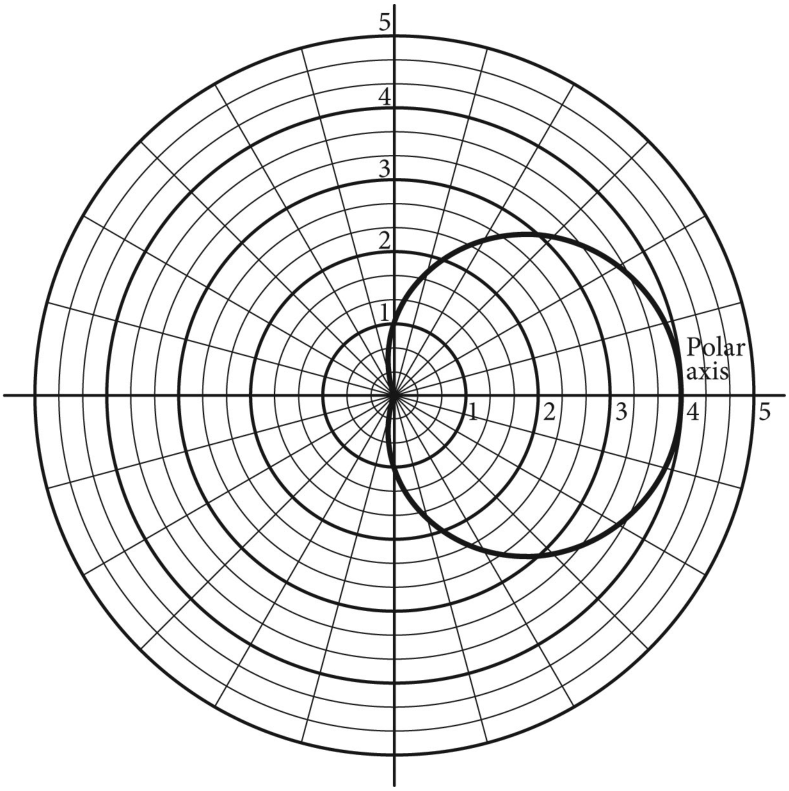
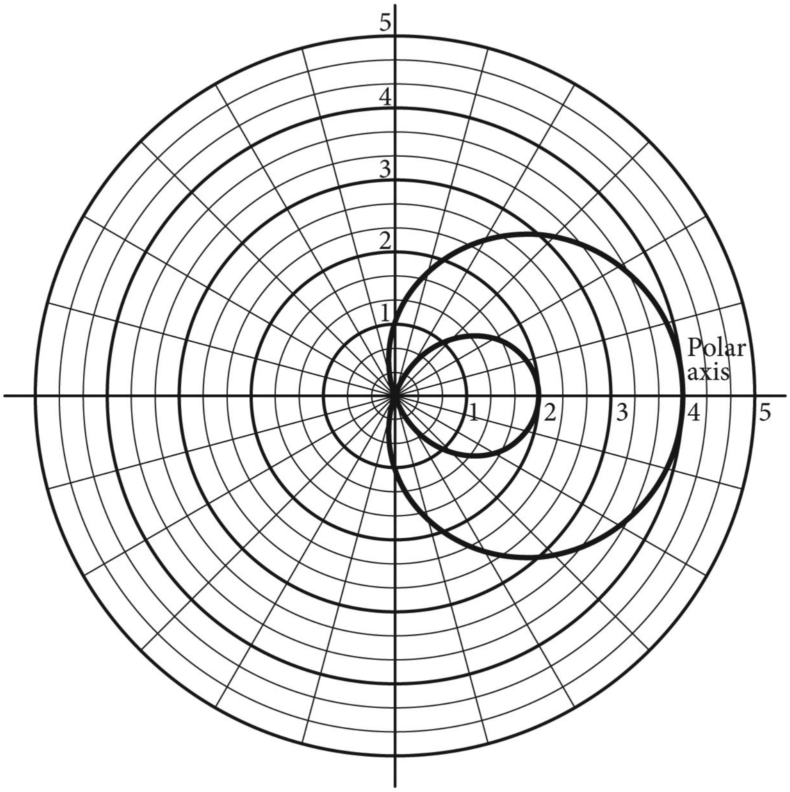
17. Which of the following is the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3\cos\theta + 2$, in the polar coordinate system for $0 \le \theta \le 2\pi$?

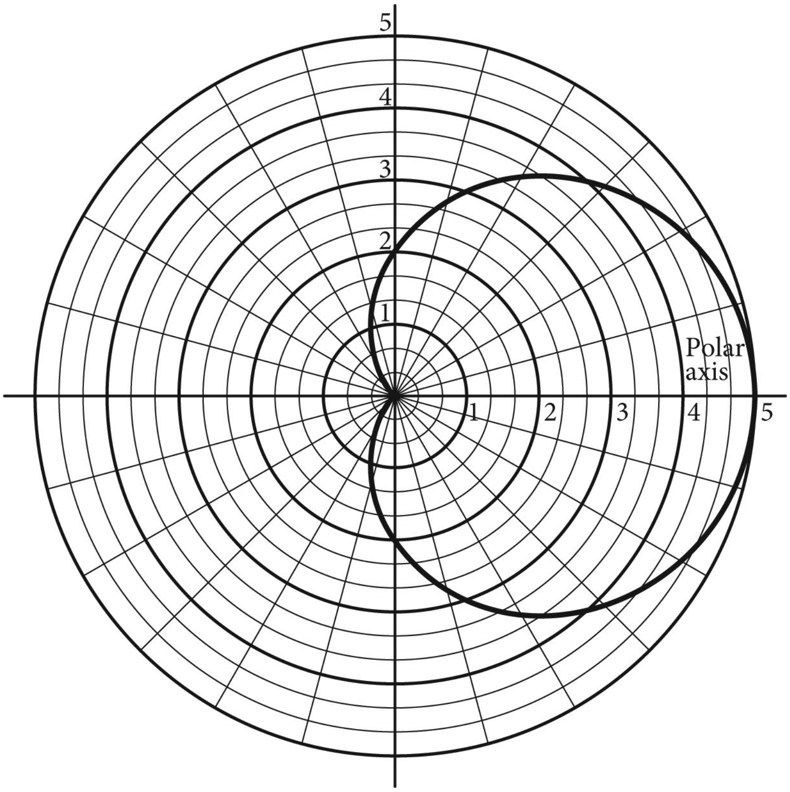
(A)



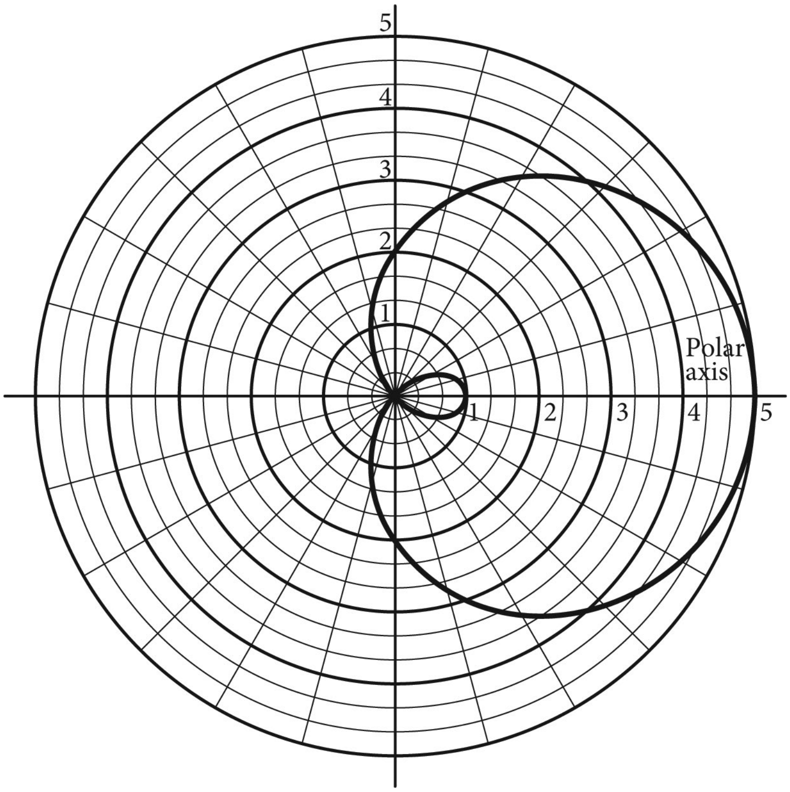
(B)



(C)



(D)



18. What are all values of $\theta, -\pi \le \theta \le \pi$, for which $2\cos\theta > -1$ and $2\sin\theta > \sqrt{3}$?

(A) $-\frac{5\pi}{6} < \theta < \frac{5\pi}{6}$

(B) $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$ only

(C) $-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$ only

(D) $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ only

19. A polar function is given by $r = f(\theta) = -1 + \sin\theta$. As θ increases on the interval $0 < \theta < \frac{\pi}{2}$, which of the following is true about the points on the graph of $r = f(\theta)$ in the xy-plane?

(A) The points on the graph are above the x-axis and are getting closer to the origin.

(B) The points on the graph are above the x-axis and are getting farther from the origin.

(C) The points on the graph are below the x-axis and are getting closer to the origin.

(D) The points on the graph are below the x-axis and are getting farther from the origin.

PART B

A graphing calculator is required for some questions on this part of the exam.

20. The temperature, in degrees Celsius (°C), in a city on a particular day is modeled by the function T defined by $T(t) = \dfrac{75t^3 - 836t^2 + 3100t - 4185}{14t^2 + 10t - 35}$, where t is measured in hours from 12 P.M. for $2 \le t \le 9$. Based on the model, how many hours did it take for the temperature to increase from $0°\text{C}$ to $5°\text{C}$?

(A) 7.701

(B) 5.420

(C) 4.114

(D) 2.280

x	$f(x)$
−2	10
−1	15
1	40
2	56

21. The table presents values for a function f at selected values of x. An exponential regression $y = ab^x$ is used to model these data. What is the value of $f(1.5)$ predicted by the exponential function model?

(A) 46.767

(B) 47.342

(C) 47.800

(D) 47.917

22. The number of minutes of daylight per day for a certain city can be modeled by the function D given by $D(t) = 160\cos\left(\frac{2\pi}{365}(t - 172)\right) + 729$, where t is the day of the year for $1 \le t \le 365$. Which of the following best describes the behavior of $D(t)$ on day 150?

(A) The number of minutes of daylight per day is increasing at a decreasing rate.

(B) The number of minutes of daylight per day is decreasing at a decreasing rate.

(C) The number of minutes of daylight per day is increasing at an increasing rate.

(D) The number of minutes of daylight per day is decreasing at an increasing rate.

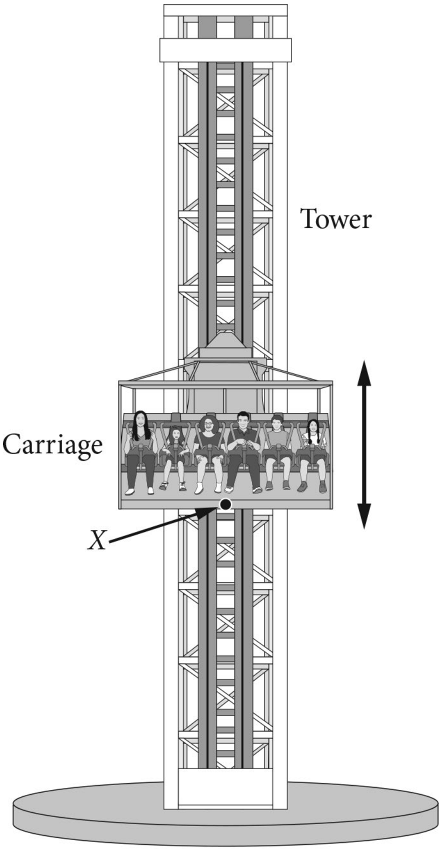
23. The function g is given by $g(x) = \sin x - \cos x$ and has a period of 2π. In order to define the inverse function of g, which of the following specifies a restricted domain for g and provides a rationale for why g is invertible on that domain?

(A) $0 \le x \le \pi$, because all possible values of $g(x)$ occur without repeating on this interval.

(B) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$, because all possible values of $g(x)$ occur without repeating on this interval.

(C) $0 \le x \le \pi$, because the length of this interval is half of the period.

(D) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$, because the length of this interval is half of the period.



Note: Figure not drawn to scale.

24. A theme park thrill ride involves a tower and a carriage that rapidly moves passengers up and down along a vertical axis, as shown in the figure. The carriage is lifted to the top of the tower, then released to move down the tower. The ride involves 10 controlled bounces from the highest point to the lowest point, and back to the highest point. A point X is located on the bottom of the carriage. The height of X above the ground, in feet, can be modeled by a periodic function H. At time $t = 0$ seconds, X is at its highest point of 120 feet. The lowest point for X is at a height of 20 feet. The next time X is at its highest point is at time $t = 8$ seconds, which is the end of the first bounce. Which of the following can be an expression for $H(t)$, where t is the time in seconds?

(A) $50\sin\left(\frac{\pi}{4}t\right) + 70$

(B) $50\cos\left(\frac{\pi}{4}t\right) + 70$

(C) $50\sin\left(\frac{\pi}{8}t\right) + 70$

(D) $50\cos\left(\frac{\pi}{8}t\right) + 70$

Section II: Free-Response

The following are examples of the kinds of free-response questions found on the exam.

PART A

A graphing calculator is required for these questions.

x	1	2	3	4	5
$f(x)$	−10	−5	4	17	34

1. Let f be an increasing function defined for $x \geq 0$. The table gives values for $f(x)$ at selected values of x. The function g is given by $g(x) = \dfrac{x^3 - 14x - 27}{x + 2}$.

   A. i. The function h is defined by $h(x) = (g \circ f)(x) = g(f(x))$. Find the value of $h(5)$ as a decimal approximation, or indicate that it is not defined. Show the work that leads to your answer.

   ii. Find the value of $f^{-1}(4)$, or indicate that it is not defined.

   B. i. Find all values of x, as decimal approximations, for which $g(x) = 3$, or indicate there are no such values.

   ii. Determine the end behavior of g as x decreases without bound. Express your answer using the mathematical notation of a limit.

   C. i. Based on the table, which of the following function types best models function f: linear, quadratic, exponential, or logarithmic?

   ii. Give a reason for your answer in part C (i) based on the relationship between the change in the output values of f and the change in the input values of f. Refer to the values in the table in your reasoning.

2. Students who completed a class participated in a year-long study to see how much content from the class they retained over the following year. At the end of the class, students completed an initial test to determine the group's content knowledge. At that time ($t = 0$), the group of students achieved a score of 75 out of 100 points. For the next 12 months, the group was evaluated at the end of each month to track their retention of the content. After 3 months ($t = 3$), the group's score was 70.84 points.

   The group's score can be modeled by the function R given by $R(t) = a + b\ln(t + 1)$, where $R(t)$ is the score, in points, for month t, and t is the number of months since the initial test.

   A. i. Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $R(t)$.

   ii. Find the values for a and b as decimal approximations.

   B. i. Use the given data to find the average rate of change of the scores, in points per month, from $t = 0$ to $t = 3$ months. Express your answer as a decimal approximation. Show the computations that lead to your answer.

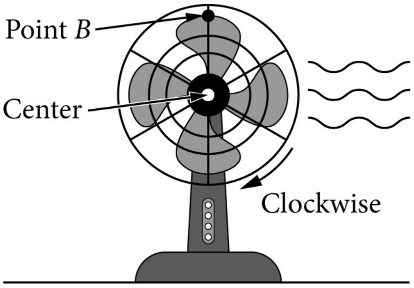
   ii. Interpret the meaning of your answer from part B (i) in the context of the problem.

   iii. Consider the average rates of change of R from $t = 3$ to $t = p$ months, where $p > 3$. Are these average rates of change less than or greater than the average rate of change from $t = 0$ to $t = 3$ months found in part B (i)? Explain your reasoning. Your explanation should include a reference to the graph of R.

   C. The leaders of the study decide to use model R to make predictions about the group's score beyond 12 months (1 year). For a given year, model R is an appropriate model if the group's predicted score at the end of the year is at least 1 point lower than the group's predicted score at the end of the previous year. Based on this information, for how many years is model R an appropriate model? Give a reason for your answer. (Note: The end of a year occurs every 12 months from the initial evaluation—$t = 12, t = 24, \ldots$)

PART B

No calculator is allowed for these questions.

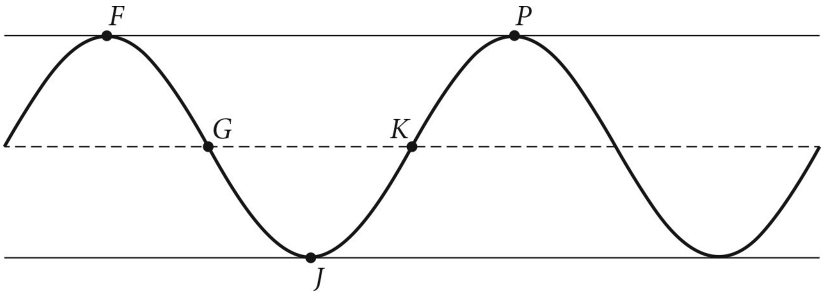


Note: Figure not drawn to scale.

3. The blades of an electric fan rotate in a clockwise direction and complete 5 rotations every second. Point B is on the tip of one of the fan blades and is located directly above the center of the fan at time $t = 0$ seconds, as indicated in the figure. Point B is 6 inches from the center of the fan. The center of the fan is 20 inches above a level table on which the fan sits. As the fan blades rotate at a constant speed, the distance between B and the surface of the table periodically decreases and increases.

   The sinusoidal function h models the distance between B and the surface of the table, in inches, as a function of time t, in seconds.

   A. The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.
   Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.



   B. The function h can be written in the form $h(t) = a\sin(b(t+c)) + d$. Find values of constants a, b, c, and d.

   C. Refer to the graph of h in part A. The t-coordinate of K is t_1, and the t-coordinate of P is t_2.

      i. On the interval (t_1, t_2), which of the following is true about h ?

         a. h is positive and increasing.
         b. h is positive and decreasing.
         c. h is negative and increasing.
         d. h is negative and decreasing.

      ii. On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

4. Directions:

- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x + 3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3 + \ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

A. The functions g and h are given by

$$g(x) = 3\ln x - \frac{1}{2}\ln x$$

$$h(x) = \frac{\sin^2 x - 1}{\cos x}.$$

i. Rewrite $g(x)$ as a single natural logarithm without negative exponents in any part of the expression. Your result should be of the form $\ln(\text{expression})$.

ii. Rewrite $h(x)$ as an expression in which $\cos x$ appears once and no other trigonometric functions are involved.

B. The functions j and k are given by

$$j(x) = 2(\sin x)(\cos x) - \cos x$$

$$k(x) = 8e^{(3x)} - e.$$

i. Solve $j(x) = 0$ for values of x in the interval $\left[0, \frac{\pi}{2}\right]$.

ii. Solve $k(x) = 3e$ for values of x in the domain of k.

C. The function m is given by

$$m(x) = \cos(2x) + 4.$$

Find all input values in the domain of m that yield an output value of $\frac{9}{2}$.

Answer Key and Question Alignment to Course Framework

Multiple-Choice Question	Answer	Skill	Learning Objective	Essential Knowledge
1	B	3.A	1.6.A	1.6.A.3
2	A	2.A	1.1.B	1.1.B.4
3	A	1.C	1.11.A	1.11.A.1
4	C	3.A	1.5.B	1.5.B.2
5	C	3.A	1.11.A	1.11.A.1
6	C	2.B	1.5.A	1.5.A.5
7	D	2.A	1.12.A	1.12.A.5
8	B	1.B	2.12.A	2.12.A.1
9	A	2.B	2.1.B	2.1.B.2
10	B	2.A	2.7.A	2.7.A.2
11	A	3.C	2.6.B	2.6.B.1
12	D	1.C	2.5.B	2.5.B.1
13	D	1.B	2.5.B	2.5.B.2
14	C	1.A	2.13.A	2.13.A.2
15	C	1.A	3.10.A	3.10.A.1
16	B	2.A	3.6.A	3.6.A.6
17	D	2.B	3.14.A	3.14.A.1
18	D	1.A	3.10.A	3.10.A.1
19	C	3.A	3.15.A	3.15.A.2
20	D	1.A	1.14.C	1.14.C.1
21	A	3.B	2.5.A	2.5.A.5
22	A	3.B	3.5.A	3.5.A.5
23	B	3.C	3.9.A	3.9.A.2
24	B	1.C	3.7.A	3.7.A.1

Free-Response Question	Skills	Learning Objectives
1	1.A, 1.C, 2.A, 3.A, 3.C	1.7.A, 1.8.A, 1.13.A, 2.7.A, 2.8.A, 2.8.B
2	1.B, 1.C, 3.B, 3.C	1.2.A, 1.13.B, 1.14.C, 2.14.A
3	1.C, 2.A, 2.B, 3.A	1.1.A, 1.1.B, 1.14.C, 3.6.A, 3.7.A
4	1.A, 1.B	2.12.A, 2.13.A, 3.10.A, 3.12.A, 3.12.C

AP PRECALCULUS

Scoring Guidelines

Scoring Guidelines for Question 1: Function Concepts
Part A: Graphing calculator required — 6 points

x	1	2	3	4	5
$f(x)$	-10	-5	4	17	34

Let f be an increasing function defined for $x \geq 0$. The table gives values for $f(x)$ at selected values of x. The function g is given by $g(x) = \dfrac{x^3 - 14x - 27}{x + 2}$.

Model Solution	Scoring	
A. i. The function h is defined by $h(x) = (g \circ f)(x) = g(f(x))$. Find the value of $h(5)$ as a decimal approximation, or indicate that it is not defined. Show the work that leads to your answer. ii. Find the value of $f^{-1}(4)$, or indicate that it is not defined.		
i. $h(5) = g(f(5)) = g(34) = \dfrac{(34)^3 - 14(34) - 27}{34 + 2} = 1077.806$	Value	**1 point** 2.A
ii. Because f is increasing on its domain, f^{-1} exists. From the table, $f^{-1}(4) = 3$.	Value	**1 point** 2.A
	Total for part (A)	**2 points**
B. i. Find all values of x, as decimal approximations, for which $g(x) = 3$, or indicate there are no such values. ii. Determine the end behavior of g as x decreases without bound. Express your answer using the mathematical notation of a limit.		
i. $g(x) = 3 \rightarrow \dfrac{x^3 - 14x - 27}{x + 2} = 3$ $x = 4.875$	Answer	**1 point** 1.A
ii. As x decreases without bound, eventually $g(x)$ increases without bound. Therefore, $\lim\limits_{x \to -\infty} g(x) = \infty$.	End behavior with limit notation	**1 point** 3.A
	Total for part (B)	**2 points**

C. i. Based on the table, which of the following function types best models function f: linear, quadratic, exponential, or logarithmic?

ii. Give a reason for your answer in part C (i) based on the relationship between the change in the output values of f and the change in the input values of f. Refer to the values in the table in your reasoning.

i. f is best modeled by a quadratic function.

Answer **1 point**

1.C

ii.

x	1st Differences in $f(x)$-values	2nd Differences in $f(x)$-values
1		
2	$-5-(-10)=5$	
3	$4-(-5)=9$	$9-5=4$
4	$17-4=13$	$13-9=4$
5	$34-17=17$	$17-13=4$

Because the 2nd differences in the output values are a constant 4 over consecutive equal-length input-value intervals, a quadratic model is best.

Reason **1 point**

3.C

Total for part (C) **2 points**

Total for Question 1 **6 points**

Scoring Guidelines for Question 2: Modeling a Non-Periodic Context
Part A: Graphing calculator required 6 points

Students who completed a class participated in a year-long study to see how much content from the class they retained over the following year. At the end of the class, students completed an initial test to determine the group's content knowledge. At that time ($t = 0$), the group of students achieved a score of 75 out of 100 points. For the next 12 months, the group was evaluated at the end of each month to track their retention of the content. After 3 months ($t = 3$), the group's score was 70.84 points.

The group's score can be modeled by the function R given by $R(t) = a + b\ln(t+1)$, where $R(t)$ is the score, in points, for month t, and t is the number of months since the initial test.

Model Solution **Scoring**

A. i. Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $R(t)$.

ii. Find the values for a and b as decimal approximations.

Model Solution	Scoring	
i. Because $R(0) = 75$ and $R(3) = 70.84$, two equations to find a and b are $a + b\ln(0+1) = 75$ $a + b\ln(3+1) = 70.84.$	Two equations	**1 point** 1.C
ii. $a = 75$ $b = -3.000806$ $R(t) = 75 - 3.001\ln(t+1)$	Values of a and b	**1 point** 1.C

Total for part (A) **2 points**

B. i. Use the given data to find the average rate of change of the scores, in points per month, from $t = 0$ to $t = 3$ months. Express your answer as a decimal approximation. Show the computations that lead to your answer.

ii. Interpret the meaning of your answer from part B (i) in the context of the problem.

iii. Consider the average rates of change of R from $t = 3$ to $t = p$ months, where $p > 3$. Are these average rates of change less than or greater than the average rate of change from $t = 0$ to $t = 3$ months found in part B (i)? Explain your reasoning. Your explanation should include a reference to the graph of R.

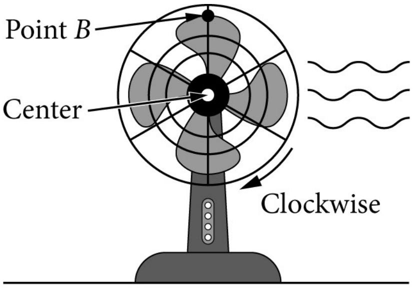
i. $\dfrac{R(3) - R(0)}{3 - 0} = -1.387$ The average rate of change is -1.387 points per month.	Average rate of change	**1 point** 1.B
ii. From $t = 0$ to $t = 3$ months, on average, the group's score was decreasing at a rate of 1.387 points per month.	Interpretation	**1 point** 3.B
iii. The average rates of change are found by using secant lines that pass through the points $(3, R(3))$ and $(p, R(p))$, where $p > 3$. Because R is logarithmic and decreasing, its graph is concave up. These secant lines are above the graph of R, and the slopes of the lines are increasing as t increases. Therefore, the average rates of change of R, in points per month, from $t = 3$ to $t = p$ months are greater than the average rate of change from $t = 0$ to $t = 3$ months.	Answer with explanation	**1 point** 3.C

Total for part (B) **3 points**

C. The leaders of the study decide to use model R to make predictions about the group's score beyond 12 months (1 year). For a given year, model R is an appropriate model if the group's predicted score at the end of the year is at least 1 point lower than the group's predicted score at the end of the previous year. Based on this information, for how many years is model R an appropriate model? Give a reason for your answer. (Note: The end of a year occurs every 12 months from the initial evaluation—$t = 12, t = 24, \ldots$)

$\lvert R(24) - R(12) \rvert \geq 1$ $\lvert R(36) - R(24) \rvert \geq 1$ $\lvert R(48) - R(36) \rvert < 1$ R is decreasing, and the graph of R is concave up. Therefore, year 4 is the first year for which the model is not appropriate. Model R is appropriate for the first 3 years ($t = 0$ to $t = 36$).	Answer with reason	**1 point** **3.C**
	Total for part (C)	**1 point**
	Total for Question 2	**6 points**

Scoring Guidelines for Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed **6 points**



Note: Figure not drawn to scale.

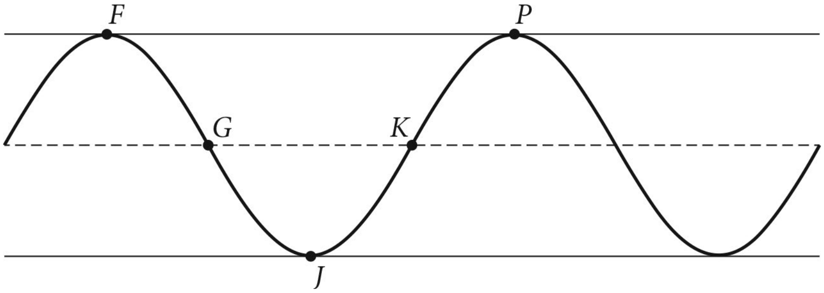
The blades of an electric fan rotate in a clockwise direction and complete 5 rotations every second. Point B is on the tip of one of the fan blades and is located directly above the center of the fan at time $t = 0$ seconds, as indicated in the figure. Point B is 6 inches from the center of the fan. The center of the fan is 20 inches above a level table on which the fan sits. As the fan blades rotate at a constant speed, the distance between B and the surface of the table periodically decreases and increases.

The sinusoidal function h models the distance between B and the surface of the table, in inches, as a function of time t, in seconds.

Model Solution **Scoring**

A. The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.

Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.



F has coordinates (0, 26).

G has coordinates (0.05, 20).

J has coordinates (0.1, 14).

K has coordinates (0.15, 20).

P has coordinates (0.2, 26).

Note: *t*-coordinates will vary. A correct set of coordinates for one full cycle of *h* as pictured is acceptable.

$h(t)$-coordinates	**1 point** 2.B
t-coordinates	**1 point** 2.B
Total for part (A)	**2 points**

B. The function h can be written in the form $h(t) = a\sin(b(t+c)) + d$. Find values of constants a, b, c, and d.

$h(t) = a\sin(b(t+c)) + d$ $a = 6$ $b = \frac{2\pi}{0.2} = 10\pi$ $c = 0.05$ $d = 20$ $h(t) = 6\sin(10\pi(t+0.05)) + 20$ Note: Based on horizontal shifts and reflections, there are other correct forms for $h(t)$.	Vertical transformations: Values of a and d	**1 point** **1.C**
	Horizontal transformations: Values of b and c	**1 point** **1.C**

Total for part (B) **2 points**

C. Refer to the graph of h in part A. The t-coordinate of K is t_1, and the t-coordinate of P is t_2.

i. On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.
b. h is positive and decreasing.
c. h is negative and increasing.
d. h is negative and decreasing.

ii. On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

i. Choice a.	Function behavior	**1 point** **2.A**
ii. The graph of h is concave down on the interval (t_1, t_2), and the rate of change of h is decreasing on the interval (t_1, t_2).	Concavity of graph and behavior of rate of change	**1 point** **3.A**

Total for part (C) **2 points**

Total for Question 3 **6 points**

Scoring Guidelines for Question 4: Symbolic Manipulations
Part B: Graphing calculator not allowed **6 points**

Directions:

- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x + 3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3 + \ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

Model Solution	Scoring	

A. The functions g and h are given by

$$g(x) = 3\ln x - \frac{1}{2}\ln x$$
$$h(x) = \frac{\sin^2 x - 1}{\cos x}.$$

i. Rewrite $g(x)$ as a single natural logarithm without negative exponents in any part of the expression. Your result should be of the form ln(expression).

ii. Rewrite $h(x)$ as an expression in which $\cos x$ appears once and no other trigonometric functions are involved.

Model Solution	Scoring	
i. $g(x) = 3\ln x - \frac{1}{2}\ln x$ $g(x) = \frac{5}{2}\ln x$ $g(x) = \ln\left(x^{5/2}\right), x > 0$ –OR– $g(x) = 3\ln x - \frac{1}{2}\ln x$ $g(x) = \ln x^3 - \ln x^{1/2}$ $g(x) = \ln\left(\frac{x^3}{x^{1/2}}\right)$ $g(x) = \ln\left(x^{5/2}\right), x > 0$	Expression for $g(x)$	**1 point** **1.B**
ii. $h(x) = \frac{\sin^2 x - 1}{\cos x}$ $h(x) = \frac{-\cos^2 x}{\cos x}$ $h(x) = -\cos x$, $\cos x \neq 0$	Expression for $h(x)$	**1 point** **1.B**
	Total for part (A)	**2 points**

B. The functions j and k are given by

$$j(x) = 2(\sin x)(\cos x) - \cos x$$

$$k(x) = 8e^{(3x)} - e.$$

i. Solve $j(x) = 0$ for values of x in the interval $\left[0, \frac{\pi}{2}\right]$.

ii. Solve $k(x) = 3e$ for values of x in the domain of k.

i. $2(\sin x)(\cos x) - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

Solutions to $j(x) = 0$ **1 point** **1.A**

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ii. $8e^{(3x)} - e = 3e$

$8e^{(3x)} = 4e$

$e^{(3x)} = \frac{e}{2}$

$\ln\left(e^{(3x)}\right) = \ln\left(\frac{e}{2}\right)$

$3x = \ln\left(\frac{e}{2}\right)$

$x = \frac{1}{3}\ln\left(\frac{e}{2}\right)$

$x = \frac{\ln e - \ln 2}{3}$

$x = \frac{1 - \ln 2}{3}$

Solution to $k(x) = 3e$ **1 point** **1.A**

Total for part (B) **2 points**

C. The function m is given by

$$m(x) = \cos(2x) + 4.$$

Find all input values in the domain of m that yield an output value of $\frac{9}{2}$.

$m(x) = \frac{9}{2} \Rightarrow \cos(2x) + 4 = \frac{9}{2}$

Values of θ in $0 \le \theta \le 2\pi$ with $\cos\theta = \frac{1}{2}$ **1 point** **1.A**

$\cos(2x) = \frac{1}{2}$

$2x = \frac{\pi}{3} + 2\pi n$ or $2x = \frac{5\pi}{3} + 2\pi n$

$x = \frac{\pi}{6} + \pi n$ or $x = \frac{5\pi}{6} + \pi n$, where n is any integer.

General solution expression **1 point** **1.A**

Total for part (C) **2 points**

Total for Question 4 **6 points**

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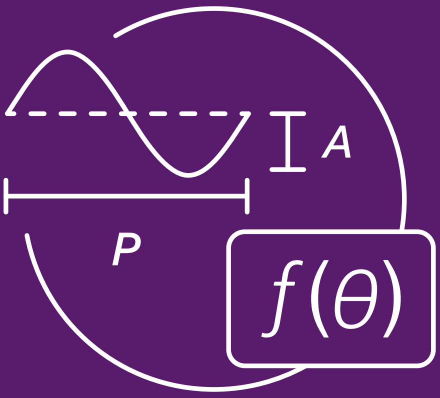
AP PRECALCULUS

Appendix



A
P
f(θ)

AP Precalculus Learning Notes

Learning Notes

UNIT 1

While studying polynomials, students use the zeros or roots or x-intercepts of the graph of a polynomial function to connect the graphical, numerical, analytical, and verbal representations. The lessons learned with polynomials apply to rational functions and extend to include a deeper introduction to the relationship between zero and infinity. When investigating the vertical, horizontal, and slant asymptotes of graphs, students engage with the zero-infinity duality that occurs often in both mathematical and contextual phenomena. As the values of one variable increase without bound, infinitely, the values of the other variable become indistinguishable, zero distance, from a limiting value. All phenomena that are inversely proportional demonstrate this zero-infinity limit behavior.

Students need to think about functions as representing how quantities co-vary. Over equal-length input-value intervals, they should consider how values of the output variable change with respect to values of the input variable and consider whether the rates of change are increasing or decreasing. Throughout the course, students should practice communicating with this language.

Teaching polynomial and rational functions should be more than familiarizing students with a set of rules and procedures. While the processes associated with finding zeros, determining equations of asymptotes, and curve sketching are included in this unit, time should be taken to build the underlying ideas. Students should also communicate how graphical and numerical representations are related to analytical representations. Polynomials provide opportunities to talk about concavity and estimate points of inflection of graphs and are also useful in developing language around relative and absolute extrema, a topic critical to calculus. While studying rational functions, students should practice using the language of, and wrestling with, the concept of limit as they explore vertical, horizontal, and slant asymptotes. For example, a student may say, “As x approaches 2 from the left, the function g decreases without bound, so the graph of g has a vertical asymptote at $x = 2$,” and “As x increases without bound, the function g has a limit of 1, so the graph of g has a horizontal asymptote at $y = 1$.” Students should be encouraged to communicate these important ideas verbally in class, as well as in writing through a variety of processing activities. They should also practice explaining the relevance of their statements in that context. As students develop polynomial and rational function models for contextual scenarios, they should be making conclusions with related numerical values and units situated in the context.

Polynomial functions will be used to model aspects of a scenario. Polynomials are especially useful in building models with multiple constraints. Phenomena that increase and decrease repeatedly without a predictable, periodic pattern benefit from polynomial modeling. Every time the degree of a polynomial increases, an extra turn or twist may be added to the graph of the polynomial function. Consequently, polynomials can model the height above the ground versus distance from the starting point of a roller coaster. The full power of polynomials will not be experienced until the study of calculus, when students learn that polynomials, a type of function that is “algebraic” and of which we have a great understanding, can be used to approximate many “transcendental” functions, of which we have far less ability to manipulate and control, to any degree of accuracy desired. So polynomial functions are a critical building block for advanced work in STEM fields.

In this unit, students begin modeling aspects of contextual scenarios and data sets through transformations applied to parent functions and technology-based regressions. Students will evaluate a situation or data set for indications as to which is the appropriate function type to use as a model, choose an appropriate function type, model the situation or data set with an appropriate function model, and make conclusions about the scenario using the model. In addition to choosing, developing, and using a model, students will add assumptions and restrictions to the models they develop. Students need to ask questions of each other that help them restrict their model, clarify assumptions related to the model, and defend their choice of a certain model. This type of interaction with other students will prepare them for modeling throughout the course.

UNIT 2

In precalculus courses, students typically learn how to manipulate exponential and logarithmic functions and equations. However, in AP Precalculus, students also develop conceptual understanding of how exponential and logarithmic functions change, how to model real-world phenomena with these functions, and what multiple representations of these functions reveal about each other. Students also hone their thinking about zero and infinity from Unit 1, as exponential and logarithmic functions demonstrate asymptotic behavior. Developing a comprehension beyond procedural knowledge helps students to develop a deeper understanding of exponential and logarithmic functions.

Although most students entering precalculus have experience working with linear and exponential functions, they frequently struggle to see the parallels between them. In this unit, students should discover the similarities between discrete arithmetic sequences and continuous linear functions, as well as discrete geometric sequences and continuous exponential functions. Students should also explore the similarities between linear and exponential functions. Since repeated addition can be understood as multiplication and repeated multiplication can be understood as exponentiation, students should identify the parallels between the linear form $f(x) = b + mx$ and the exponential form $f(x) = ab^x$. Both forms include an initial value and a constant involved with change and students should recognize that the only structural difference is the operation of addition versus multiplication. Note that it is useful to move students from traditional slope-intercept form $y = mx + b$ to $f(x) = b + mx$ for these activities. As students explore these relationships, they should experience a multirepresentational approach to arithmetic and geometric sequences and linear and exponential functions. Making connections among the graphical, numerical, analytical, and verbal function representations allows for a fuller understanding.

As a transition from exponential to logarithmic functions, students first study logarithmic expressions alone and begin to think in terms of a logarithmic scale. Next, students will develop an understanding of composition of functions and inverses in the setting of general (non-analytical) functions such as f and g before moving to the more analytical experiences with exponential functions and their inverses, logarithmic functions. Students should experience composition of functions with different function representations and develop communication skills related to function composition. For example, students may be given a table of values for the function f and a graph for g and be asked to evaluate $g(f(2))$. They may then communicate that "The table for f maps 2 to 7, then the graph for g maps 7 to approximately 4. So, the composition maps 2 to 4." Students should experience a composite function being undefined at certain values because the output from the first function is not in the domain of the second function. Students should also experience graphing a composition of two functions, with and without technology, which includes holes, places where the composition is not defined, and asymptotes. As students reverse the order of the functions, they will find that function composition is not commutative.

The relationship between composition of functions and inverses should be highlighted in this unit. Students should practice composing functions with the identity function $f(x) = x$ analytically, graphically, and numerically, and, additionally, find that composition with $f(x) = x$ does not change the other function. Students should experience the parallels between $f(x) = x$ as the identity function for composition just as 0 is the identity for addition and 1 is the identity for multiplication. This development with "identity" will allow for a smoother transition to working with inverse functions because additive inverses add to give 0, multiplicative inverses multiply to give 1, and function inverses compose to give x. Because an inverse function acts as the reverse mapping of a function, students should practice verbally explaining the operations mapping values of x to values of $y = f(x)$, then work together to explain how the inverse function f^{-1} maps values of y to $x = f^{-1}(y)$. These relationships can be leveraged to show that $x = f^{-1}(y) = f^{-1}(f(x))$. That is, the composition of two functions that are inverses of each other maps to the identity function $f(x) = x$. There may be a temptation to just apply the "swap x and y then solve for y" approach to inverses. However, this approach can lead to misconceptions about inverse functions including confusion of the roles and meanings of the independent and dependent variables for a function and its inverse, and errors in thinking about and labeling the axes when graphing a function and its inverse. This procedure should be treated with great care as it invokes a complete reversal of the roles and assignment of the variables, including their respective meanings and units, when applicable. Instruction should focus on the conceptual development of inverse functions so that students are able to build models, explain relationships among function representations, and provide rationales for inverse relationships. With this strong conceptual development of inverse functions, the graphical, numerical, and analytical relationships between exponential functions and logarithmic functions have a firm foundation for relating the function types and using that knowledge to solve exponential and logarithmic equations.

It is important for students to gain numerical understandings of logarithms before exploring logarithmic functions as the inverse functions of exponential functions. The use of logarithmic scales can help students see how logarithms can manage numerical data that tends towards large values. As the inverse relationship between exponential and logarithmic functions is investigated, respective rates of change should also be understood as inverses: the output values of a general form exponential function change proportionately over equal-length input-value intervals while the input values of a general form logarithmic function change proportionately over equal-length output-value intervals.

Students will also experience comparing competing function models for bivariate data sets and choosing appropriate models by taking the residuals in regression analysis into account. As students use technology to calculate linear, quadratic, cubic, quartic, exponential, and logarithmic regressions on data sets, they should not lose sight of the conceptual importance of the residual plots and what those plots communicate.

UNIT 3

In addition to the standard treatment of trigonometric functions, this unit emphasizes modeling with trigonometric functions, with and without technology. Unlike transformations on linear, quadratic, exponential, and some polynomial functions, transformations on the sine and cosine functions are, graphically speaking, very apparent. With sinusoidal functions, vertical and horizontal additive transformations – or graphical translations – and vertical and horizontal multiplicative transformations – or graphical dilations – are not only easy to see both analytically and graphically, but come with a rich vocabulary for communicating such changes including vertical shift, phase shift, midline, amplitude, and period. Students should recognize and communicate these characteristics as they build models. From the scatterplot of an appropriate data set, students should be able to estimate the amplitude, vertical shift, and period of a sinusoidal function, build a model from those values, and estimate the phase shift needed to bring the model into phase with the data.

There are also other important concepts that students will explore in this unit. Starting with right triangle trigonometry and the unit circle, students should investigate why the sine function is a wave and why the graph of the tangent function has asymptotes. Through classroom explorations and discussions, students should be using language of covarying quantities. Since the input values of trigonometric functions represent angles and the output values relate to arcs of a circle, students should practice explaining how the values change with respect to each other and how the rates of change are changing. For example, a student may say "From π over 2 to π radians, as the angle increases the values of sine decrease, and over equal-angle intervals, the values of sine decrease at a decreasing rate." Practicing communicating changing rate of rates in this context is productive preparation for calculus.

Students will solve trigonometric equations, work with trigonometric identities, and determine exact values of trigonometric functions for certain angles. While they solve equations, students should account for periodicity and domain restrictions. Domain restrictions related to inverse trigonometric functions require special attention and students should be given an opportunity to restrict the domain before the standard restrictions are given to them. These skills are important for working with trigonometric functions in calculus and other college-level STEM courses.

Students will also explore a new coordinate system. Polar coordinates describe a location in the plane by a distance from a reference point and an angle from a reference direction. Trigonometry is key in relating the familiar rectangular coordinate system to the polar coordinate system. While studying polar functions, students should engage with covarying quantities in this novel setting. The set of input values for a polar function are angles and the output values are radii. Consequently, intervals of increase or decrease, maxima and minima, and rates of change take on different meanings that are related to characteristics of functions in the rectangular coordinate system. Students should be able to communicate that "As the angle increases from π over 2 to π the radius starts at 4, increases to 6, decreases to 0, and then increases again." The transition in thinking to a different coordinate system forces students to revisit their thinking about function behavior—zeros, extrema, and especially relative rates of change. Just as studying a different language deepens understanding of one's native language, so can the study of functions based on a different coordinate system strengthen students' understanding of functions, their behavior, and their characteristics.

UNIT 4

Since most of the content in this unit is likely to be unfamiliar to students, it will be important for them to have contextual anchors and language on which to build their understandings. When someone throws a ball to someone else, the ball's path is a parabola. Through Algebra 1, Algebra 2, and earlier units in this course, this path would be described in terms of height and distance from the first person as $y = f(x)$, a quadratic function of x. Parametric functions can also describe the same path, but as two equations based on the parameter of time. The vertical height of the ball from the ground at any time t seconds after it is thrown is given by $y = v(t)$, and the horizontal distance from the first person is given by $x = h(t)$. With this parametric function of vertical height and horizontal distance both as functions of time, the two components can be analyzed individually and with respect to each other. Furthermore, rates of change of individual components with respect to changes in values in the input variable can also be understood independently or in relation to one another.

Alternately, using the particle motion in the plane context, students should express statements about both position and motion in a direction. They should also be justifying their conclusions while making claims. For example, a student may say "The particle is moving up and to the left from time t equals 1 to t equals 2 seconds because y of t is increasing on that interval and x of t is decreasing," or "The particle is farthest to the right at time t equals 3.5 seconds because x of t has a maximum at that time," or "The only time the particle is at the origin is at time t equals 4 seconds because x of t equals zero only at times 1, 3, and 4 seconds; y of t equals zero only at times 2, 4, and 6 seconds; and t equals 4 seconds is the only time when x of t and y of t both equal zero."

For many students, this will be the first time they graph parametric functions. Technology should be used as a tool to explore these graphs. However, students should not just look at the graph produced by technology, but they should also be looking at tables of values to better understand how the parameter t impacts each dependent variable x and y. The concept of domain is again critical in bounding parametric functions. Students should experience the same mathematical object expressed by two different parametric functions over different domains. In addition, it is important for students to understand that graphs of parametric functions may not pass the "vertical line test." This more advanced understanding of a function will stretch those students who had memorized a rule or procedure without internalizing the uniqueness requirement of a function.

Students will learn how to graph implicitly-defined functions and examine how variables vary with respect to each other. Conic sections can be expressed by implicitly-defined functions and can be analyzed and graphed in this form. Conic sections and all other implicitly-defined functions can also be parameterized or rewritten as parametric functions. The topics of parametric functions, implicitly-defined functions, and the familiar shapes of conic sections help students to understand the relationships in meaningful ways. In understanding conic sections parametrically, students strengthen their understanding of trigonometric functions and trigonometric identities since the tie between implicit and parametric expressions of conic sections relies on trigonometry.

After a brief introduction to vectors and matrices, students should begin to experience linear transformations via matrices and vectors in concrete ways. Students may begin with vectors representing the vertices of a random quadrilateral in Quadrant I of the xy-plane and apply a 90-degree rotation matrix to see how the matrix affects the quadrilateral. Students can discover the impacts of various matrix transformations through exploratory activities or guided instruction. In the end, students should be able to produce three matrices that, when applied to a large set of vectors, rotate, dilate, and reflect an object in the xy-plane.

As students begin modeling situations involving changes in states, care should be taken to compare the matrix operation and transformation results to calculations done by hand to help students get a feel for how the matrices are interacting with the situation being modeled. Students should practice communicating the results of applying different matrices to a given vector.

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