Teacher Guide for AP Calculus

2018-19

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Welcome

Dear Colleagues:

The 2018 implementation of new Advanced Placement Program® classroom resources and annual processes is built on years of conversations with AP educators, who have the best understanding of the rewards and opportunities of offering challenging college-level coursework in high school. We believe that teachers deserve more effective resources, yearlong support, and meaningful feedback to help students develop the skills they need for success on the AP Exam, in college, and beyond.

An important component of the 2018 implementation is the creation of new resources for teachers and students. These resources have been designed to build student understanding over time through multiple exposures to concepts and incremental development of practices and skills. As part of your participation in the 2018 implementation of new AP classroom resources and annual processes, you will have access to selected resources before they launch worldwide in 2019-20.

On behalf of the College Board and the Advanced Placement Program, thank you for being a part of the 2018 implementation. As you implement the resources, we look forward to hearing from you and continuing to develop features that best serve teachers and students.

Teaching AP Calculus By Design

AP Calculus is organized into units that address the scope and sequence of the course and the three constructs students must develop throughout the year:

- Big Ideas
- Skills Listed in Instructional Planning Report (IPR)
- Mathematical Practices for AP Calculus (MPACs)

The resources have been developed to provide a cohesive, unified experience throughout the course. The resources are divided into 8 units:

- **Unit 1:** Limits and Continuity
- **Unit 2:** Derivatives
- **Unit 3:** Existence Theorems
- **Unit 4:** Using Derivatives to Analyze Functions
- **Unit 5:** Applications of Derivatives
- **Unit 6:** Accumulation and Riemann Sums
- **Unit 7:** Antiderivatives and the Fundamental Theorem of Calculus
- **Unit 8:** Applications of Definite Integrals

Big Ideas: AP Calculus is structured around three foundational concepts: limits, derivatives, and integrals and the Fundamental Theorem of Calculus. Throughout the course, students should have opportunities to engage with these concepts so that by the end of the course, students can demonstrate proficiency in understandings related to each concept.

Big Ideas	**Enduring Understandings**		
Big Idea 1: Limits			
The idea of limits is essential for discovering and developing important ideas, definitions, formulas, and theorems in calculus.	EU 1.1: The concept of a limit can be used to understand the behavior of functions.	EU 1.2: Continuity is a key property of functions that is defined using limits.	
Big Idea 2: Derivatives			
Using derivatives to describe the rate of change of one variable with respect to another variable allows students to understand change in a variety of contexts.	EU 2.1: The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies.	EU 2.2: A function's derivative, which is itself a function, can be used to understand the behavior of the function. EU 2.3: The derivative has multiple interpretations and applications including those that involve instantaneous rates of change.	EU 2.4: The Mean Value Theorem connects the behavior of a differentiable function over an interval to the behavior of the derivative of that function at a particular point in the interval.
Big Idea 3: Integrals and the Fundamental Theorem of Calculus			
Integrals are used in a wide variety of practical and theoretical applications.	EU 3.1: Antidifferentiation is the inverse process of differentiation. EU 3.2: The definite integral of a function over an integral is the limit of a Riemann sum over that interval and can be calculated using a variety of strategies.	EU 3.3: The Fundamental Theorem of Calculus, which has two distinct formulations, connects differentiation and integration. EU 3.4: The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation.	EU 3.5: Antidifferentiation is an underlying concept involved in solving separable differential equations. Solving separable differential equations involves determining a function or relation given its rate of change.

Skills Listed in Instructional Planning Report (IPR): Students in AP Calculus must learn a set of specific skills related to course content. Throughout the course, students should have opportunities to practice each of the skills. By the end of the course, students need to be able to demonstrate proficiency in each aspect of the skills and in a variety of contexts. Student performance on each skill is provided in the IPR following the exam administration.

The skills are:

- Approximate values and functions
- Select and apply procedures for limits and derivatives
- Select and apply procedures for integrals
- Establish conditions for definitions and theorems
- Justify properties and behaviors of functions
- Interpreting context
- Analyzing problems in context
- Interpreting notational expression

Mathematical Practices for AP Calculus (MPACs): The MPACs articulate the behaviors in which students need to engage in order to achieve conceptual understanding in AP Calculus. Each concept and topic addressed in the course can be linked to one or more of the MPACs. The MPACs are intended to be utilized frequently within units across the course in diverse contexts.

- **MPAC 1: Reasoning with definitions and theorems**
  Students can:
  a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;
  b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;
  c. apply definitions and theorems in the process of solving a problem;
  d. interpret quantifiers in definitions and theorems (e.g., "for all," "there exists");
  e. develop conjectures based on exploration with technology; and
  f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

- **MPAC 2: Connecting concepts**
  Students can:
  a. relate the concept of a limit to all aspects of calculus;
  b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;
  c. connect concepts to their visual representations with and without technology; and
  d. identify a common underlying structure in problems involving different contextual situations.

- **MPAC 3: Implementing algebraic/computational processes**
  Students can:
  a. select appropriate mathematical strategies;
  b. sequence algebraic/computational procedures logically;
  c. complete algebraic/computational processes correctly;
  d. apply technology strategically to solve problems;
  e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and
  f. connect the results of algebraic/computational processes to the question asked.

- **MPAC 4: Connecting multiple representations**
  Students can:
  a. associate tables, graphs, and symbolic representations of functions;
  b. develop concepts using graphical, symbolical, verbal, or numerical representations with and without technology;
  c. identify how mathematical characteristics of functions are related in different representations;
  d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);
  e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and
  f. consider multiple representations (graphical, numerical, analytical, and verbal) of a function to select or construct a useful representation for solving a problem.

- **MPAC 5: Building notational fluency**
  Students can:
  a. know and use a variety of notations;
  b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);
  c. connect notation to different representations (graphical, numerical, analytical, and verbal); and
  d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

- **MPAC 6: Communicating**
  Students can:
  a. clearly present methods, reasoning, justifications, and conclusions;
  b. use accurate and precise language and notation;
  c. explain the meaning of expressions, notation, and results in terms of a context (including units);
  d. explain the connections among concepts;
  e. critically interpret and accurately report information provided by technology; and
  f. analyze, evaluate, and compare the reasoning of others.

Implementing AP Resources

The resources created for AP Calculus are designed to be directly implemented into the course curriculum.

Dashboard
shows class and student performance for individual assignments and unit tests

Teacher Guide and Unit Guides
include a course curriculum map that provides suggested pacing and sequence for course content and skills

> **Topics**
> describe instructional focus for unit lessons
>
> **Focus Topics**
> focus on and scaffold specific content and skill development

Teacher Modules
online professional development resources including AP test scoring information, sample questions, instructional activities, and other resources

Lesson Plans and Focus Quizzes
tie to the focus topics and let students practice the skills that are so important in this course

test students' understanding of challenging and foundational or important concepts

Question Bank
access to a bank of released summative AP exam questions, plus new questions aligned to the AP Calculus units

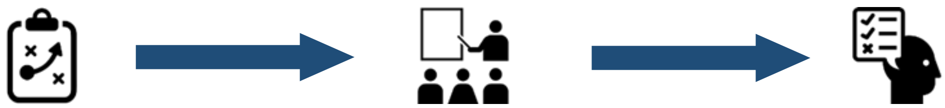
Unit Tests
formative, scaffolded questions that test students' performance on the key content and skills outlined in the beginning of each unit

Links to Other College Board Resources
connections to classroom resources available on AP Central, Summer 2018 APSI materials, and where to go for more information in the Teacher Modules

Using this Guide

Integrate the Calculus resources throughout the course to help students develop disciplinary practices, reasoning skills, and thematic understandings. Follow the instructional model outlined below to incorporate the numerous resources into your classroom.

Instructional Model



TEACHERS

PLAN

Prior to teaching each unit, use the **Unit Guide** to consider how you will focus your instruction.

- Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- Identify the **Instructional Approaches** section in the CED that will be useful in teaching the focus practices and skills
- Watch the **Teacher Module(s)** for the topic

TEACH

As you teach each unit, refer to the instructional sequence in the unit guide for details on how the resources align to the AP course framework.

- Teach each lesson. Emphasize the **focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- Use provided **lesson plans** to reinforce skill development and conceptual understanding
- Remember that **focus topics are critical**, and that students should master these skills and this content
- Assign **Focus Quizzes** to check for understanding

ASSESS

Measure student proficiency with the content and skills covered in the unit and provide feedback to students.

- Assign the **Teacher Module Practice Question** and give feedback to students before they take the Unit Test
- Target the **scoring focus in the Unit Test** as you apply the rubric to student work
- Create additional practice opportunities using the **AP Question Bank** and assign them through Fine Tune

STUDENTS

FOCUS

- Provide opportunities to learn and practice by scaffolding and sequencing **skills**
- Teach **learning objectives** in the context, and building toward, **Enduring Understanding**

PRACTICE

- Practice applying practices and skills in context using **lesson plans**
- Receive feedback after completing **Focus Quizzes**

DEMONSTRATE

- Complete **Unit Test** to demonstrate understanding and skill mastery

Scope and Sequence of Skills and Content

The following pages provide a visual overview of the learning progression throughout the year.

The Learning Progression

	Unit 1 Limits and Continuity	Unit 2 Derivatives	Unit 3 Existence Theorems	Unit 4 Using Derivatives to Analyze Functions
Teaching Time (45-60 min. periods)	**14–21 class periods**	**21–28 class periods**	**5–10 class periods**	**10–15 class periods**
FOCUS Target the skill covered in the **Teacher Module**	***Limits: Approximating Values and Functions***	***Selecting Procedures for Limits and Derivatives***	***Establishing Conditions for Definitions and Theorems***	***Justifying Properties and Behaviors of Functions***
TEACH Within each unit are multiple **focus topics**, each of which addresses a piece of challenging content tied to a skill and one or more big idea. Use **resources for focus topics**, which include lesson plans and focus quizzes. Big ideas across the unit focus topics: **Limits** **Derivatives** **Integrals**	**Approximating Limit Values from a Graph** **Focus Skill** **Approximating Values and Functions** **Using Tables to Approximate Limit Values** **Focus Skill** **Approximating Values and Functions** **Selecting Procedures for Calculating Limits** **Focus Skill** **Selecting and Applying Procedures**	**Applying the Chain Rule for Derivatives of Composite Functions** **Focus Skill** **Selecting and Applying Procedures** **Categorizing Functions for Derivative Rules** **Focus Skill** **Selecting and Applying Procedures**	**Establishing Continuity for EVT and IVT** **Focus Skill** **Establishing Conditions for Definitions and Theorems** **Establishing Differentiability for the MVT** **Focus Skill** **Establishing Conditions for Definitions and Theorems**	**Using the First Derivative to Justify Properties and Behaviors of Functions** **Focus Skill** **Justifying Properties and Behaviors of Functions** **Second Derivative Test** **Focus Skill** **Justifying Properties and Behaviors of Functions**
ASSESS Assign the **unit test**, which contains multiple-choice and free-response questions.	**11 Multiple-Choice Questions** **FRQ 1: Estimate limits of functions from both graphs and tables** **FRQ 2: Interpret and use limit notation and select procedures for calculating limits**	**11 Multiple-Choice Questions** **FRQ 1: Categorize functions for derivative rules** **FRQ 2: Apply the chain rule for derivatives of composite functions**	**11 Multiple-Choice Questions** **FRQ 1: Establish continuity for applying EVT and IVT** **FRQ 2: Establish differentiability for applying MVT**	**11 Multiple-Choice Questions** **FRQ 1: Use the first derivative to justify properties and behaviors or functions** **FRQ 2: Use the second derivative to justify properties and behaviors of functions.**

Unit 5	Unit 6	Unit 7	Unit 8
Applications of Derivatives	Accumulation and Riemann Sums	Antiderivatives and the Fundamental Theorem of Calculus	Applications of Definite Integrals
8–13 class periods	**7–12 class periods**	**16–21 class periods**	**11–16 class periods**
Analyzing Problems in Context	*Definite Integrals: Interpreting Notational Expressions*	*Applying Procedures for Integration by Substitution*	*Interpreting Context for Definite Integrals*
Analyzing Problems Involving Rates of Change in Applied Contexts Focus Skill Analyzing Problems in Context	Approximating a Definite Integral Using Midpoint and Trapezoidal Sums Focus Skill Approximating Values and Functions	U-Substitution: Applying Procedures for Integrals Focus Skill U-Substitution: Applying Procedures for Integrals	Accumulation and Net Change in Context Focus Skill Interpreting Context
Analyzing Problems Involving Related Rates Focus Skill Analyzing Problems in Context	Interpreting Summation Notation Focus Skill Interpreting Notational Expressions	Interpreting Behavior of $f(x)$ from a Graph of $f'(x)$ Focus Skill Interpreting Behavior of $f(x)$ from a Graph of $f'(x)$	Analyzing Problems Involving Definite Integrals and Accumulation Focus Skill Analyzing Problems in Context
		Applying Procedures for Separable Differential Equations and General Solutions Focus Skill Applying Procedures for Separable Differential Equations and General Solutions	Analyzing Problems Involving Integrals and Motion Focus Skill Analyzing Problems in Context
11 Multiple-Choice Questions FRQ 1: Analyze problems involving optimization and using the First Derivative Test to determine max. and min. values FRQ 2: Analyze problems involving rates of change in applied contexts	11 Multiple-Choice Questions FRQ 1: Interpret symbolic expressions, especially those involving summation notation FRQ 2: Interpret definite integral notation, and connect it to summation notation	11 Multiple-Choice Questions FRQ 1: Apply procedures for evaluating integrals, focusing on u-substitution FRQ 2: Apply procedures for solving separable differential equations	11 Multiple-Choice Questions FRQ 1: Apply definite integrals to problems involving area and volume FRQ 2: Apply definite integrals to problems involving motion, accumulation, and next change in context

AP Calculus		Unit
Limits and Continuity	**Suggested Length:** 14-21 class periods	**1**

PLAN

- ☐ Review the **Curriculum Framework** in the *AP Calculus Course and Exam Description* (CED)
- ☐ Identify **Instructional Approaches** in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Limits: Approximating Values and Functions*

OVERVIEW/CONTEXT

- An understanding of limits is essential for discovering and developing important ideas, definitions, formulas, and theorems in calculus. While students must have procedural understanding of how to compute the values of limits, even more important is a solid, conceptual understanding of limits, which helps them to be successful in Unit 1 and subsequent units in the course.
- Students should be able to work with various mathematical representations, including tables, graphs, and analytic expressions, when determining the value of a limit of a function as the function approaches a point or approaches infinity.
- Students should know and understand the algebraic properties of limits, techniques for finding limits of indeterminate forms, and behavior of a function near a point, as well as the relationship between limits and the continuity of a function.
- Teachers may want to consider forgoing a formal review of pre-calculus topics and hone specific skills and content as they become relevant throughout the course.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 4: Connecting multiple representations *Extract and interpret mathematical content from any presentation of a function* Students should be able to connect limits expressed analytically to the same limits expressed by graph or by table.	• Instructional Strategy: *Create representations* (CED p. 33) • Teacher Module: *Limits: Approximating Values and Functions* – Student Handout: *Estimating Limits by Creating Different Representations* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 74-76)
MPAC 5: Building notational fluency *Connection notation to different representations (graphical, numerical, analytical, and verbal)* Students should be able to articulate what a limit actually represents, when given a graph, a table, a verbal expression, or an analytic function.	• Instructional Strategy: *Notation read-aloud* (CED p. 36) • APSI Lesson 1: *Goals of AP Calculus* (p. 9) • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 77-79)
MPAC 3: Implementing algebraic/computational processes *Complete algebraic/computational processes correctly* Students should be able to determine the limit of an analytic function by completing the algebraic steps required from a selected procedure.	• Instructional Strategy: *Error analysis* (CED p. 34) • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 71-73)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Approximating Values and Functions *Using tables and graphs to approximate limit values* Students should understand that calculus is more than just "solving problems". Students need to be able to estimate limits from a variety of presentations, including graphs and tables.	• Teacher Module: *Limits: Approximating Values and Functions* – Video: *Teaching the Skill of Approximating Values and Functions with Challenging Topics: Limits* • Lesson Plan and Student Handout: *Approximating Limit Values from a Graph* • *Special Focus: Approximation* https://secure-media.collegeboard.org/apc/ap-sf-calculus-approximation.pdf
Selecting and Applying Procedures *Selecting procedures for determining limits from analytic functions* Students need to practice selecting procedures for finding limits of functions, not just being able to apply given procedures and carry out algebraic steps.	• Lesson Plan and Student Handout: *Selecting Procedures for Calculating Limits*

Instructional Emphasis for Learning Objectives	Resources
LO 1.1A(b) Express limits symbolically using correct notation Students should be able to use the correct notation when determining limits analytically, or from information provided in a graph or table. **LO 1.1B Estimate limits of functions** Students should be able to estimate limits when provided information in different representations, such as a graph or table.	• Concept Outline (CED p. 11) • APSI Lesson 4: *Understanding the Big Ideas* (p. 36)
LO 1.1C Determine limits of functions Students should be able to find the limits of functions analytically using algebraic manipulation, alternate forms of trigonometric functions, and basic theorems.	• Concept Outline (CED p. 12) • APSI Lesson 4: *Understanding the Big Ideas* (p. 36)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE

Focus Topic **Focus Quiz Available** **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
0	Introduction to Calculus	--	--	Students should know what calculus is (e.g., looking at something that is infinitely small, or infinitely many), how the course is generally broken up into limits, derivatives, and integrals, what types of things calculus is used for,.	--
1	Defining a Limit	1.1A(b) 1.1B	• Given a function *f*, the limit of *f(x)* as *x* approaches *c* is a real number *R* if *f(x)* can be made arbitrarily close to *R* by taking *x* sufficiently close to *c*, but not equal to *c*. (EK 1.1A1) • Numerical and graphical information can be used to estimate limits. (EK 1.1B1)	Students should be able to explain what is meant by "infinitely close" when discussing limits. They should understand how to use and interpret basic limit notation, and also master the idea that a limit exists when a function approaches the same value from both the left and the right. Students should also be able to determine when a limit does not exist.	Interpreting Notational Expressions
2 (Focus Topic; Focus Quiz Available; Lesson Plan Available)	Approximating Limit Values from a Graph	1.1A(b) 1.1B	• Given a function *f*, the limit of *f(x)* as *x* approaches *c* is a real number *R* if *f(x)* can be made arbitrarily close to *R* by taking *x* sufficiently close to *c*, but not equal to *c*. (EK 1.1A1) • Numerical and graphical information can be used to estimate limits. (EK 1.1B1)	Students should be able to use graphs to directly connect limits and limit notation to the behavior of functions. For example, graphs help students better understand limits at infinity and the concept of asymptotes as end behavior (as opposed to the common misconception of "lines that the graph cannot cross").	Approximating Values and Functions
3 (Focus Topic; Focus Quiz Available)	Using Tables to Approximate Limit Values	1.1A(b) 1.1B	• Given a function *f*, the limit of *f(x)* as *x* approaches *c* is a real number *R* if *f(x)* can be made arbitrarily close to *R* by taking *x* sufficiently close to *c*, but not equal to *c*. (EK 1.1A1) • Numerical and graphical information can be used to estimate limits. (EK 1.1B1)	Students should be able to use tables to understand that limits are what a function approaches, not what it equals. Tables also provide an accessible format for students to see limits from the left and limits from the right before this concept is formally introduced.	Approximating Values and Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Interpreting Graphical Behavior for One-Sided Limits	1.1A(b) 1.1D	• A limit might not exist for some functions at particular values of *x*. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right. (EK 1.1A3) • Asymptotic and unbounded behavior of functions can be explained and described using limits. (EK 1.1D1)	Students should be able to write one-sided limits based on graphical representations and tables, and from a given one-sided limit select an appropriate graphical representation (and understand the reasoning for that selection). *Key point*: A one-sided limit can exist even if it doesn't equal the limit from the other side.	Justifying Properties and Behaviors of Functions
5	Connecting Limits and Graphical Behavior	1.1A(a) 1.1A(b)	• A limit might not exist for some functions at particular values of *x*. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right. (EK 1.1A3) • Numerical and graphical information can be used to estimate limits. (EK 1.1B1)	Students should be able to write limit notation from information provided graphically, and produce a graphical representation from a given notational limit expression. Students have already looked at graphs and estimating limits, but here the focus will be on writing limit expressions when given a graph, and on sketching a graph when given a limit statement. *Key point*: Limit statements provide key information about the behavior of a function, but note that there's more than one way a graph could be sketched based on a particular limit expressions.	Interpreting Notational Expressions
6	Defining Continuity	1.2A	• A function *f* is continuous at *x=c* provided that *f(c)* exists, the limit of *f* as *x* approaches *c* exists, and that limit is equal to *f(c)*. (EK 1.2A1) • Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous at all points in their domains. (EK 1.2A2)	Students should be able to discuss the continuity of specific functions and understand what it means for a function to be continuous, given graphs or given an algebraic function. Teachers should encourage discussion of the continuity of functions on a closed interval.	Justifying Properties and Behaviors of Functions
7	Limit Rules for Sums, Differences, Products, Quotients, and Composite Functions	1.1C	• (Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules. (EK 1.1C1)	Students should be able to use general rules for finding the limits of various functions: sums, products, quotients, and composite functions.	Selecting and Applying Procedures
8	Determining Limits Using Direct Substitution	1.1C	• Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules. (EK 1.1C1)	Students should be able to find the limits of specific functions using direct substitution, and determine when direct substitution cannot be used.	Selecting and Applying Procedures

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
9	Determining Limits by Factoring	1.1C	• The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the Squeeze Theorem. (EK 1.1C2)	Students should be able to recognize that, in cases when direct substitution results in a fraction of the form 0/0 (which is called an indeterminate form), they might be able to use algebraic manipulation to find the limit. One of these methods is factoring and canceling common factors. Students should be able to explain why this method works. *Key point*: Students should remember that the reduced fraction is not the same as the original function, and that a graphing calculator usually will not display a point discontinuity.	Selecting and Applying Procedures
10	Determining Limits Using Conjugates	1.1C	• The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem. (EK 1.1C2)	Students should be able to use the technique of finding limits with conjugates, which also involves a review of what a conjugate is and how to multiply them.	Selecting and Applying Procedures
11	Determining Trigonometric Limits	1.1C	• The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem. (EK 1.1C2)	Students should be able to find the limits of trigonometric functions, like y = sin *x* and y = cos *x*, which are continuous everywhere. They should also be able to find the limits of other functions, such as y = tan *x*, as well as know when and how to use the Squeeze Theorem.	Selecting and Applying Procedures
12	Infinite Limits	1.1A(a) 1.1A(b) 1.1D	• The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits. (EK 1.1A2) • Asymptotic and unbounded behavior of functions can be explained and described using limits. (EK 1.1D1)	Students should be able to recognize infinite limits, connect limits to the concept of asymptotes, and connect infinite limits and all previous limits to graphical representations. Students should be reminded that a limit only exists if the function approaches the same value from both the left and the right. If the function approaches infinity from the left and the right, then the limit does not exist.	Interpreting Notational Expressions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
13	Analyzing Functions for Discontinuities	1.2A	• A function *f* is continuous at *x=c* provided that *f(c)* exists, the limit of *f* as *x* approaches *c* exists, and that limit is equal to *f(c)*. (EK 1.2A1) • Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes. (EK 1.2A3)	Students should be able to analyze functions for discontinuities, including discontinuities that can be "fixed". They also should be able to determine specific *x*-values where functions are not continuous, and also categorize the type of discontinuity. Students should be able to examine piecewise-defined functions with an unknown variable (e.g. *a*) and solve for the value of a that would make the function continuous on the given interval.	Justifying Properties and Behaviors of Functions
14	Selecting Procedures for Calculating Limits	1.1C	• Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules. (EK 1.1C1) • The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem. (EK 1.1C2) • Limits of indeterminate forms may be evaluated using L'Hospital's Rule. (EK 1.1C3)	Students should be able to explain why an incorrect procedure is not chosen just as well as they can explain why the correct procedure is chosen. Teachers should focus on unpacking the decision-making process and concepts students should be considering when deciding which procedure to use.	Selecting and Applying Procedures

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 1 Test
- ☐ Target the **scoring focus** for Unit 1 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 1 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Limits: Approximating Values and Functions*

Practice Question Description	Scoring Focus for Unit 1
Approximate Values and Functions Free-Response Question Part (a): Students use data in a table to estimate the value of a limit at a specific point in time. *Note: Parts (b), (c), and (d) involve content from other units.*	Part (a): Students must be able to work with both analytic and tabular representations of functions and limits. Focus should be placed on student ability to understand limits given a variety of representations.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 1
FRQ #1	Focus should be placed on ensuring students can estimate limits of functions from both graphs and tables. Students should also be able to: • Identify key and relevant information needed to solve a problem • Extract and interpret mathematical content from information presented graphically, analytically, and numerically • Assess the reasonableness of results or solutions
FRQ #2	Focus should be placed on ensuring students can interpret and use limit notation and select procedures for calculating limits. Students should also be able to: • Compute accurate results or solutions, sequencing procedures logically • Use accurate mathematical notation • Present precise conclusions that answer the questions asked

AP Calculus

Unit 2

Derivatives

Suggested Length:
21-28 class periods

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Selecting Procedures for Limits and Derivatives*

OVERVIEW/CONTEXT

- Using derivatives to describe the rate of change of one variable with respect to another variable allows students to understand change in a variety of contexts.
- Students build their understanding of the derivative using the concept of limits, and use the derivative primarily to computer the instantaneous rate of change of a function.
- In this unit, students should be able to use different definitions of the derivative, estimate derivatives from tables and graphs, and apply various derivative rules and properties.
- While emphasis in this unit is usually placed on mastering the procedures related to applying different derivative rules, teachers should place an increased focus on *selecting* the correct procedures and ensure that students can articulate why specific rules are more appropriate for particular functions.
- Teachers should also make sure that students do not lose sight of the meaning behind derivative rules – including how calculations lead to the slopes of tangent lines and instantaneous rates of change.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 3: Implementing algebraic/computational processes *Select appropriate mathematical strategies* Students should be able to select the derivative rule appropriate for different types of functions.	• Teacher Module: *Selecting Procedures for Derivatives* – Student Handout: *Flowchart: Selecting a Procedure for Derivatives* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 71-73) • APSI Lesson 8: *Unpacking the Mathematical Practices* (p. 91-94)
MPAC 3: Implementing algebraic/computational processes *Complete algebraic/computational processes correctly* Students should be able to correctly apply the procedures for derivative rules.	• Instructional Strategy: *Discussion groups* (CED p. 34) • Teacher Module: *Selecting Procedures for Derivatives* – Student Handout: *Which Procedure and Why?* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 71-73)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Selecting and Applying Procedures *Selecting procedures for calculating derivatives* The emphasis in this unit is on *selection*: students need explicit practice with selecting derivative procedures for different types of functions, and understanding why certain selections are more appropriate than others.	• Teacher Module: *Selecting Procedures for Derivatives* – Video: *Teaching the Skill of Selecting Procedures with Challenging Topics: Derivatives* • Lesson Plan and Student Handout: *Categorizing Functions for Derivative Rules*

Instructional Emphasis for Learning Objectives	Resources
LO 2.1C Calculate derivatives Students should be able to calculate derivatives using rules for different classes of functions, including but not limited to: sums, differences, products, quotients, and composite functions.	• Lesson Plan and Student Handout: *Applying the Chain Rule for Derivatives of Composite Functions*

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE — **Focus Topic** — **Focus Quiz Available** — **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1	Secant Lines, Average Rate of Change, and Defining Instantaneous Rate of Change	2.1A	• The instantaneous rate of change of a function at a point can be expressed by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$. (EK 2.1A2)	Students should have a graphical/intuitive idea of what an instantaneous rate of change is, leading to a beginning idea of what a tangent line is. A review of slopes and rates of change may be necessary when using secant lines to begin to define the instantaneous rate of change.	Interpreting Notational Expressions
2	Using Technology to Estimate Derivatives; Derivatives from Tables	2.1A 2.1B	• The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval. (EK 2.1A1) • The derivative can be represented graphically, numerically, analytically, and verbally (EK 2.1A5) • The derivative at a point can be estimated from information given in tables or graphs. (EK 2.1B1)	Students should be able to use the slope of the secant line to estimate the derivative, when knowing general values of a function around a point. Students should understand that this information can be given in tabular form, and that technology can be used to examine how a function behaves at specific points. Using the definition of the derivative and the graphing calculator, students can estimate the derivative of a function with a high degree of accuracy, regardless of the function given. With the idea of secant lines, we can estimate the derivative near a certain point, and identify that quantity as an estimate of the instantaneous rate of change.	Approximating Values and Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
3	The Limit Definition of the Derivative, and Practice	2.1A	• The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval. (EK 2.1A1) • The instantaneous rate of change of a function at a point can be expressed by $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$, provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$. (EK 2.1A2)	Students should know the formal limit definition of the derivative and its alternate form. They should also use approximations from previous lessons of simple polynomials and compare with the values obtained by using the formal definition of the derivative. Students should learn that this limit definition provides a precise method for calculating these simple polynomial limits. This lesson also helps connect the idea of a limit (i.e. what the function is approaching, not what it equals) with the secant line processes used in earlier lessons.	Interpreting Notational Expressions
4	Continued Practice of Definition of the Derivative; Notation and Equations of Tangent Lines	2.1A 2.3B	• For $y = f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y'. (EK 2.1A4) • The derivative at a point is the slope of the line tangent to a graph at that point on the graph. (EK 2.3B1)	Students should be able to make the connection between the formal definition of the derivative and the slope of the tangent line. Students should use the definition of the derivative to find equations of tangent lines to points on given functions, and graph the functions and tangent lines to see if the equations make sense.	Interpreting Notational Expressions
5	Tangent Lines and Local Linearity; When Derivatives Do Not Exist	2.3B	• The tangent line is the graph of a locally linear approximation of the function near the point of tangency. (EK 2.3B2)	Students should be able to understand the concept of local linearity – that functions tend to look like their tangent lines when viewed "close up". Students should use technology to explore this concept.	Analyzing Problems in Context
6	Connecting Differentiability and Continuity	2.2B	• A continuous function may fail to be differentiable at a point in its domain. (EK 2.2B1) • If a function is differentiable at a point, then it is continuous at that point. (EK 2.2B2)	Students should understand that all differentiable functions are continuous, and the foundational connection between differentiability and continuity. Using the graphing calculator or other technology, students should learn idea of a vertical tangent line and the graphical implications of a derivative that goes to +/- infinity (vertical asymptote).	Establishing Conditions for Definitions and Theorems

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
7	Calculating Derivatives: by Definition, Leading to the Generalized Power Rule	2.1C	• Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions. (EK 2.1C1)	Students can look at functions to see if they can find a pattern for the derivative of $y = x^n$ (where n is a positive integer) when applying the definition of the derivative. Connection should then be made to the Power Rule.	Selecting and Applying Procedures
8	Calculating Derivatives: by Power Rule, and Introducing the Sum and Difference Rules	2.1C	• Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric. (EK 2.1C2) • Sums, differences, products, and quotients of functions can be differentiated using derivative rules. (EK 2.1C3)	Students should use the Power Rule on a variety of functions, including constants, monomials, and polynomials that involve sums or differences, checking with the definition of the derivative. Students should also be able to determine when the Power Rule cannot be applied – such as functions that contain the product of two terms.	Selecting and Applying Procedures
9	Derivatives of Trigonometric Functions	2.1C	• Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions. (EK 2.1C1) • Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric. (EK 2.1C2)	Students should know how to find the derivatives of trigonometric functions, using a combination of the definition of the derivative and/or technology and exploration. Students should remember trigonometric identities from previous courses and some of the limit values from Unit 1.	Selecting and Applying Procedures
10	Calculating Derivatives: Product Rule	2.1C	• Sums, differences, products, and quotients of functions can be differentiated using derivative rules. (EK 2.1C3)	Students should be able to apply the Product Rule – but also know when it is appropriate to use the Product Rule. Focus should be placed on correct notation, and producing a step-by-step sequence of steps to find the derivative of a function that contains a product.	Selecting and Applying Procedures

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
11	Calculating Derivatives: Quotient Rule	2.1C	• Sums, differences, products, and quotients of functions can be differentiated using derivative rules. (EK 2.1C3)	Students should be able to apply the Quotient Rule – but also know when it is appropriate to use the Quotient Rule. Focus should be placed on correct notation, and producing a step-by-step sequence of steps to find the derivative of a function that contains a quotient.	Selecting and Applying Procedures
12	Applying the Chain Rule for Derivatives of Composite Functions	2.1C	• The chain rule provides a way to differentiate composite functions. (EK 2.1C4)	Students should understand function notation and composition of functions before they are introduced to the Chain Rule. They should be able to identify which function is the "outer function" and which function is the "inner function", which should lead to deeper understanding of how and why the Chain Rule works.	Selecting and Applying Procedures
13	Categorizing Functions for Derivative Rules	2.1C	• Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions. (EK 2.1C1) • Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric. (EK 2.1C2) • Sums, differences, products, and quotients of functions can be differentiated using derivative rules. (EK 2.1C3) • The chain rule provides a way to differentiate composite functions. (EK 2.1C4)	Students should have the opportunity to find derivatives when a procedure is not given, and teachers should place an emphasis on the selection process – noting what features of the problem indicate the type of rule that should be applied. This lesson is not simply giving students problems and making them take derivatives. It targets strategy and specific, common misunderstandings, like applying the product rule when a function is simply a multiple of a power function.	Selecting and Applying Procedures
14	Calculating Derivatives: Chain Rule with Other Rules, and Multiple Uses of the Chain Rule	2.1C	• The chain rule provides a way to differentiate composite functions. (EK 2.1C4)	Students should be able to recognize the need for the Chain Rule within other rules, such as the Product Rule or the Quotient Rule, as well as cases of the Chain Rule within the Chain Rule.	Selecting and Applying Procedures

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
15	Higher-Order Derivatives	2.1D	• Differentiating f' produces the second derivative f'' provided the derivative of f' exists; repeating this process produces higher order derivatives of f. (EK 2.1D1) • Higher order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and y''. Higher order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$. (EK 2.1D2)	Students should be able to use previously learned differentiation techniques to calculate second derivatives, as well as other higher order derivatives. The notation here is important, and teachers should make sure that students can use and understand notation correctly in context.	Selecting and Applying Procedures
16	Implicit Derivatives – Part I	2.1C	• The chain rule is the basis for implicit differentiation. (EK 2.1C5)	Students should focus on taking derivatives of functions of *x* and *y* with respect to *y*. Starting with an equation like *xy* = 1 (which can be separated into a "*y* =" form) can introduce the idea, and then students can work with equations that cannot be separated into an explicit "*y* =" form.	Selecting and Applying Procedures
17	Implicit Derivatives – Part II	2.1C	• The chain rule is the basis for implicit differentiation. (EK 2.1C5)	Students should focus on taking derivatives of functions with respect to a variable that is not *x* or *y*. Using *t* is the most common, and has easy to understand practical interpretation. However, teachers should not neglect using other variables so that students develop fluency with the notation.	Selecting and Applying Procedures
18	Derivatives of Inverse Functions	2.1C	• The chain rule can be used to find the derivative of an inverse function, provided the derivative of that function exists. (EK 2.1C6)	Students will need to remember the notation for inverse functions and not confuse it with an exponent of -1. Using the knowledge from the previous lessons on implicit derivatives, students should work with information given in tabular form to practice calculating derivatives of inverse functions.	Selecting and Applying Procedures

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
19	Derivatives of Inverse Trigonometric Functions	2.1C	• The chain rule can be used to find the derivative of an inverse function, provided the derivative of that function exists. (EK 2.1C6)	Students should be able to recognize and understand inverse trigonometric functions. It may be helpful for teachers to derive arcsin *x*, arccos *x*, and arctan *x* functions. Then, using the previously learned differentiation techniques, students can practice finding derivatives of functions in this form.	Selecting and Applying Procedures
20	Derivatives of Exponential Functions	2.1C	• Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions. (EK 2.1C1) • Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric. (EK 2.1C2)	Students should be able to find the derivative of $y = e^x$ through exploration with technology. They can also "discover" the pattern for the derivatives of $y = e^{2x}$ and $y = e^{3x}$, motivating how the Chain Rule works for exponential functions. Students sometimes lock on to the idea that "the derivative of e^x is e^x," and making sure that students understand how the Chain Rule applies is essential.	Selecting and Applying Procedures
21	Derivatives of Logarithmic Functions	2.1C	• Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions. (EK 2.1C1) • Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric. (EK 2.1C2)	Students should focus again on patterns to avoid misunderstandings – such as "the derivatives of ln(anything) is 1/(anything)," neglecting the Chain Rule.	Selecting and Applying Procedures

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 2 Test
- ☐ Target the **scoring focus** for Unit 2 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 2 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Selecting Procedures for Limits and Derivatives*

Practice Question Description	Scoring Focus for Unit 2
Selecting Procedures for Derivatives Free-Response Question Part (c): Students select the appropriate differentiation technique and then calculate the derivative at a point. *Note: Parts (a), (b), and (d) involve content from other units.*	Part (c): Students must be able to select the correct differentiation technique (first point) and then use the derivative to calculate a value (second point). Focus should be placed on recognizing that this function, presented in function notation, requires use of the Chain Rule.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 2
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students can categorize functions for derivative rules. Students should also be able to: • Select mathematical methods or procedures that efficiently use relevant information given or found to solve problems in context • Compute accurate results or solutions, sequencing procedures logically • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked • Connect numerical solutions with mathematical concepts
FRQ #2 (Non-calculator)	Focus should be placed on ensuring students can apply the chain rule for derivatives of composite functions. Students should also be able to: • Select mathematical methods or procedures that efficiently use relevant information given or found to solve problems in context • Compute accurate results or solutions, sequencing procedures logically • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked

AP Calculus		**Unit**
Existence Theorems	**Suggested Length:** 5-10 class periods	**3**

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Establishing Conditions for Definitions and Theorems*

OVERVIEW/CONTEXT

- In this unit, students should be able to determine the applicability of important calculus theorems using continuity and differentiability. While students may be familiar with the definition of the word "theorem," they may need guidance understanding the nuances of an existence theorem and the logic behind the applicability of these theorems.
- Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem.
- Students should understand that the Mean Value Theorem connects the behavior of a differentiable function over an interval to the behavior of the derivative of that function at a particular point in the interval.
- Several student misunderstandings arise in this unit, which is why these theorems are covered on their own. Among other things, students frequently believe that they can cite a theorem without first establishing that the conditions for the theorem have been met, and may also not recognize that differentiability implies continuity.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 1: Reasoning with definitions and theorems *Confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem* Students should be able to determine whether the conditions of continuity and differentiability have been met when given a situation involving one of the existence theorems.	• Instructional Strategy: *Graphic Organizer* (CED p. 35) • Teacher Module: *Establishing Conditions for Definitions and Theorems* – Teacher Resource: *Flow Chart for Existence Theorems* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 65-67) • APSI Lesson 12: *Strategies for Teaching AP Calculus* (p. 136-137)

MPAC 1: Reasoning with definitions and theorems
Use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results
Students should be able to construct an argument establishing that certain conditions have been met and concluding that a particular existence theorem thus applies to a given situation. Instruction should focus on reminding students that citing a definition is not the same as justifying their conclusion – that is, arguments must be tailored so that they apply to the specific problem or a given context.
	• Instructional Strategy: *Constructing an Argument* (CED p. 33)
	• Teacher Module: *Establishing Conditions for Definitions and Theorems*
	  – Student Handout: *Using Graphic Organizers to Construct an Argument*
	  – Teacher Resource: *Graphic Organizer and Template for MVT*
	• APSI Lesson 14: *Selecting Resources to Support Teaching AP Calculus* (p. 160-163)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Establishing Conditions for Definitions and Theorems *Establishing continuity for EVT and IVT* Students need to be able to explicitly state whether a function is continuous on a particular closed interval, so instruction should focus on identifying where discontinuities exist on that interval and why (e.g., corners, cusps, certain piecewise-defined functions).	• Teacher Module: *Establishing Conditions for Definitions and Theorems* – Video: *Teaching the Skill of Establishing Conditions with Challenging Topics: Continuity and Differentiability* • Focus Topic Lesson Plan: *Establishing Continuity for EVT and IVT*
Establishing Conditions for Definitions and Theorems *Establishing differentiability for the MVT* Since the MVT requires both continuity on a closed interval and differentiability on an open interval, students need to be able to explicitly state whether those conditions have been met for a given situation. Emphasizing that *differentiability implies continuity* will be key.	• Teacher Module: *Establishing Conditions for Definitions and Theorems* – Video: *Teaching the Skill of Establishing Conditions with Challenging Topics: Continuity and Differentiability* – Activity Description: Mean Value Theorem: Using graphing organizers to construct an argument • Focus Topic Lesson Plan: *Establishing Differentiability for MVT*

Instructional Emphasis for Learning Objectives	Resources
LO 1.2B Determine the applicability of important calculus theorems using continuity Students should be able to determine whether the conditions for using one of the existence theorems have been met in a given situation – i.e., continuity for EVT and IVT, and both continuity and differentiability for MVT.	• Concept Outline (CED p. 12) • APSI Lesson 15: *Addressing Prerequisite Skills Throughout the Course* (p. 172, problem AB 2012 multiple choice #9)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE

Focus Topic **Focus Quiz Available** **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1	Introducing Existence Theorems: the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem	1.2B	• Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem (EK 1.2B1).	Students are first introduced to the idea of "existence theorems" – knowing whether something exists rather than making an explicit calculation using a formula. The emphasis is on the logic of these theorems, such as "if A, then B" and "if not A, then not B."	Establishing Conditions for Definitions and Theorems
2	Establishing Continuity for EVT and IVT	1.2B	• Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem (EK 1.2B1).	In this lesson, the EVT and IVT are refined and described. Students need to understand that continuity is necessary for these theorems, so they need to know how to establish continuity (or not) and how to make this justification explicit.	Establishing Conditions for Definitions and Theorems
3	Finding Specific Values of *c* with the MVT	1.2B 2.4A	• Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem (EK 1.2B1). • If a function f is continuous over the interval $[a,b]$ and differentiable over the interval (a,b), the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval (EK 2.4A1).	In this lesson students are introduced to the computational side of the MVT, namely determining the value c where the conclusion of the MVT is met (if conditions are met). This lesson should focus on the form and meaning of the MVT, using the opportunity for a review of differentiation rules and techniques.	Justifying Properties and Behaviors of Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Establishing Differentiability for the MVT	1.2B 2.4A	• Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem (EK 1.2B1). • If a function f is continuous over the interval $[a,b]$ and differentiable over the interval (a,b), the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval (EK 2.4A1).	The MVT requires differentiability on an open interval and continuity on a closed interval, so students should focus on understanding where the MVT can and cannot be applied (an absolute value function is an easy example of the latter). Focusing on types of discontinuities and when functions may not be differentiable (corners, cusps, certain piecewise-defined functions, etc.), as well as the concept that differentiability implies continuity, but not the reverse, are key.	Establishing Conditions for Definitions and Theorems

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 3 Test
- ☐ Target the **scoring focus** for Unit 3 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 3 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Establishing Conditions for Definitions and Theorems*

Practice Question Description	Scoring Focus for Unit 3
Establishing Conditions for Definitions and Theorems Free-Response Question Part (a): Students must recognize that a polynomial function is differentiable (and thus continuous) for all values, so the Mean Value Theorem can be applied. Part (b): Students must recognize that this function is also a polynomial, and therefore continuous, so the Intermediate Value Theorem can be applied. *Note: Parts (c) and (d) involve content from other units.*	Part (a): Students must establish that the conditions for applying the MVT have been met in the process of applying the MVT to the function. Part (b): Similar to part (a), students must establish that the conditions for applying the IVT have been met in the process of applying the IVT to the function.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 3
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students can establish continuity for applying the Extreme Value Theorem and the Intermediate Value Theorem. Students should also be able to: • Identify or establish claims, hypotheses, or conjectures • Select definitions and theorems to build arguments, to justify conclusions or solutions, and to prove results • Determine whether initial conditions have been satisfied in order to apply the conclusions of a theorem • Apply definitions and theorems to provide evidence for supporting claims or conjectures, or justifying conclusions or solutions • Use accurate mathematical vocabulary and notation • Connect numerical solutions with mathematical concepts

Free-Response Question	Scoring Expectation for Unit 3
FRQ #2 (Calculator)	Focus should be placed on ensuring students can establish differentiability for applying the Mean Value Theorem. Students should also be able to: • Apply technology strategically to solve problems • Develop conjectures based on exploration with technology • Select definitions and theorems to build arguments, to justify conclusions or solutions, and to prove results • Determine whether initial conditions have been satisfied in order to apply the conclusions of a theorem • Apply definitions and theorems to provide evidence for supporting claims or conjectures, or justifying conclusions or solutions • Connect numerical solutions with mathematical concepts

AP Calculus		**Unit**
Using Derivatives to Analyze Functions	**Suggested Length:** 10-15 class periods	**4**

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Justifying Properties and Behaviors of Functions*

OVERVIEW/CONTEXT

- In this unit, students will use derivatives to analyze functions. However, it is important for focus to be placed on justification – that is, using calculus-based reasoning to support a claim about the original function's behavior.
- Teachers should ensure that students use precise, clear, and specific language to communicate their findings when applying the First or Second Derivative Tests. Students should avoid phrases like "more positive" or "keeps increasing faster".
- It is important for students to have clear definitions of terms introduced in this unit, as well as how they translate to the graphs of particular functions. For example, a student should be able to define a point of inflection, but also understand that this point can occur at values other than where $f'(x) = 0$.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 2: Connecting concepts *Connect concepts to their visual representations with and without technology* Students should be able to examine a graphical representation of a function's derivative and use its visual features to support a claim about the original function's behavior.	• Instructional Strategy: *Quick Write* (CED p. 36) • Teacher Module: *Justifying Properties and Behaviors of Functions Using Derivatives* – Student Handout: *Placemat Quick Write Activity* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 69)
MPAC 1: Reasoning with definitions and theorems *Use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results* Students should be able to state what features they see in the graph of a derivative function, what conclusions that allows them to draw about the original function, and why. Instruction should emphasize the use of clear and precise language in students' justifications.	• Instructional Strategy: *Critique Reasoning* (CED p. 34) • Teacher Module: *Justifying Properties and Behaviors of Functions Using Derivatives* – Student Handout: *Quiz Trade Activity* • APSI Lesson 2: *Developing Student Understanding* (p. 19-22) • Commentary on the Instructions for the Free Response Section of the AP Calculus Exams http://apcentral.collegeboard.com/apc/public/repository/2011_Calculus_FR_Instruction_Commentary.pdf

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Justifying Properties and Behaviors of Functions *Using the first and second derivatives to justify properties and behaviors of functions* Students need to be able to cite behaviors and properties of a derivative function, such as whether the function is increasing or decreasing, or whether the function is positive or negative on a given interval. When writing a justification based on derivatives, students must check critical points *and* endpoints, use an appropriate conclusion for the derivative test being applied, and use accurate and precise language to communicate their reasoning.	• Teacher Module: *Justifying Properties and Behaviors of Functions Using Derivatives* – Video: *Teaching the Skill of Justifying Properties and Behaviors of Functions with Challenging Topics: Derivatives* – Activity Description: *Justifying Using Derivatives* • Focus Topic Lesson Plan: *Using the First Derivative to Justify Properties and Behaviors of Functions* • Focus Topic Lesson Plan: *Using the Second Derivative to Justify Properties and Behaviors of Functions* • APSI Lesson 11: *Communicating in Mathematics: Focus on MPAC 6* (p. 114-132)

Instructional Emphasis for Learning Objectives	Resources
LO 2.2A Use derivatives to analyze properties of a function Students should be able to use calculus-based reasoning to describe the behavior of a function – that is, when given the function's first or second derivative, the student can connect the properties of those derivatives to the properties of the original function.	• Concept Outline (CED p. 15) • *Special Focus: Extrema* *http://apcentral.collegeboard.com/apc/public/repository/AP_CM_Calculus_Extrema.pdf* • APSI Lesson 13: *Using Summative Items for Instruction* (p. 148-149)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE — **Focus Topic** — **Focus Quiz Available** — **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1	L'Hospital's Rule	1.1C 1.1D	• Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule (EK 1.1C3). • Relative magnitudes of functions and their rates of change can be compared using limits (EK 1.1D2).	Once students know how to differentiate, L'Hospital's Rule becomes a powerful tool for evaluating certain limits. Emphasis should be placed on checking the conditions for use of this rule before actually applying it. Students should also apply pre-calculus knowledge of relative growth rates of functions to examine end-behavior of functions.	Selecting and Applying Procedures
2	Critical Points, Introduction to Extrema, Absolute and Relative Extrema	1.2B 2.2A	• Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem (EK 1.2B1). • First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2).	Students can apply previous knowledge about maximums and minimums (e.g. the vertex of a parabola) to form a basis for a calculus understanding of extrema via critical points. Emphasis should be placed on the fact that values where a derivative equals zero or does not exist are CANDIDATES for extrema. Calculus-based reasoning must be used for justification, which will be described in more detail in the following lessons.	Justifying Properties and Behaviors of Functions
3	First Derivative Test	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1).	The first derivative test provides calculus-based reasoning for why a critical value is a relative maximum, minimum, or neither. Students should explore examples of how and why this works, as well as how the test forms the basis for a justification of a maximum or minimum. Emphasis should be placed on the fact that a sign chart is an excellent tool, but by itself it does not constitute a justification.	Justifying Properties and Behaviors of Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Using the First Derivative to Justify Properties and Behaviors of Functions	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2).	This lesson drives home the point of using the first derivative test as a justification for relative maximums and minimums. Students should see both analytical and graphical representations of functions so that students have multiple ways to understand this concept. For example, a graph of a function f' where the graph "touches" the x-axis but does not cross it (analytically, from a repeated root, for example) is an excellent example of where a first derivative equals zero but is neither a relative maximum or minimum. Such examples provide opportunities for students to practice their justifications.	Justifying Properties and Behaviors of Functions
5	Using the Candidate's Test to Justify Properties and Behaviors of Functions	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2).	Students often need to find absolute extrema as well as relative extrema, and the Candidate's Test is most often the simplest approach. Students need to remember that the "candidates" for absolute extrema are the critical values and the endpoints. Values of the function at those points must be checked in order to find absolute extrema, and failure to check the endpoints results in insufficient justification, even if the endpoints are not in fact absolute extrema.	Justifying Properties and Behaviors of Functions
6	Concavity and Inflection Points	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2).	Inflection points and concavity can be easily defined both mathematically and visually, and students should understand them in both ways. Discussion should focus on what it means to be increasing and an increasing rate versus increasing at a decreasing rate, for example, and what that looks like in a graphical representation.	Justifying Properties and Behaviors of Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
7	Second Derivative Test	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1).	The second derivative test is another method for checking relative extrema, and in some cases may be the only way to do so. Students should understand the procedure for the second derivative test, as well as the conceptual/graphical reasons why the test works. This lesson is another excellent opportunity to look at graphical representations of functions, and not just analytical representations.	Justifying Properties and Behaviors of Functions
8	Using the Second Derivative to Justify Properties and Behaviors of Functions	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2).	Like the previous lessons on the first derivative test, this lesson deepens understanding of the process and focuses on justifications using the second derivative.	Justifying Properties and Behaviors of Functions
9	Curve Sketching	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2). • Key features of the graphs of f, f', and f'' are related to one another (EK 2.2A3).	Finding x and y intercepts, symmetry, domain, range, continuity, asymptotes, differentiability, extrema, increasing/decreasing intervals, critical values, concavity, points of inflection, and infinite limits and limits at infinity helps students synthesize everything they have learned in this Unit and previous units. Students find this synthesis very challenging, so it is helpful to walk students through examples before having them try this independently.	Justifying Properties and Behaviors of Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
10	Connecting f, f', and f''	2.2A	• First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection (EK 2.2A1). • Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations (EK 2.2A2). • Key features of the graphs of f, f', and f'' are related to one another (EK 2.2A3).	While previous units have focused on graphical behavior, this lesson emphasizes the concept and brings the function, first derivative, and second derivative together. For example, given a graph of f', identifying key features of f and f'', using f'', identifying key features of f', helps students make all the necessary connections.	Justifying Properties and Behaviors of Functions

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 4 Test
- ☐ Target the **scoring focus** for Unit 4 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 4 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Justifying Properties and Behaviors of Functions*

Practice Question Description	Scoring Focus for Unit 4
Justifying Properties and Behaviors of Functions Free-Response Question *Note: Parts (a), (b), (c), and (d) involve content from other units.* However, teachers can provide the "hint" that $H'(x) = f(x)$, in which case the following would be easily applicable: Part (a): Students must connect information provided visually in the graph of f to the behavior of H to determine increasing behavior and concavity. Part (b): Students must connect information provided visually in the graph of f to the behavior of H to determine points of inflection.	Part (a): Students must identify both intervals where H is concave up and increasing (and can receive partial credit for identifying only one of those regions). Intervals without reasons or with incorrect reasons will not receive the "reason" point. Part (b): Students must identify the three points of inflection and provide a reason why. Proper and precise use of terminology is important for earning these points.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 4
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students can use the first derivative to justify properties and behaviors of functions. Students should also be able to: • Identify key and relevant information provided and needed to solve problems • Extract and interpret mathematical content from information presented graphically, numerically, analytically, and verbally to solve problems • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Compute accurate results or solutions, sequencing procedures logically • Apply definitions and theorems to provide evidence for supporting claims or conjectures, or justifying conclusions or solutions

Free-Response Question	**Scoring Expectation for Unit 4**
FRQ #2 (Calculator)	Focus should be placed on ensuring students can use the second derivative to justify properties and behaviors of functions s. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Apply technology strategically to solve problems • Develop conjectures based on exploration with technology • Accurately report information provided by technology • Connect numerical solutions with mathematical concepts

AP Calculus		Unit 5
Applications of Derivatives	**Suggested Length:** 8-13 class periods	

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Analyzing Problems in Context*

OVERVIEW/CONTEXT

- While derivatives are easily represented as the slopes of tangent lines to curves, students sometimes have trouble seeing derivatives as rates of change in other contexts. In this unit, students are introduced to contextual problems where they need to recognize situations where use of derivatives is necessary, and then apply derivative rules and properties to solve problems.
- Students will also be introduced to the problems involving related rates, optimization, and linear motion.
- This unit contains concepts that are both challenging and important – through the lessons, students will see what derivatives are "used for in real life".
- Students typically develop common misunderstandings when working to solve a problem that involves derivatives in context. They may draw incorrectly or mislabel a diagram, write an incorrect or misrepresentative equation, apply an incorrect strategy for solving the problem, or struggle with providing correct interpretations based on a solution.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 2: Connecting concepts *Identify a common underlying structure in problems involving different contextual situations* Students should be able to recognize the features of a problem that would determine which problem-solving approach would be used – i.e., whether it's an optimization problem, a related rates problem, or a problem involving rates of change in applied contexts. Each of these types of problems has a common underlying structure in terms of what information is being provided and what's being asked, so instruction should emphasize those commonalities across a variety of contexts.	• Instructional Strategy: *Marking the Text* (CED p. 35) • Teacher Module: *Related Rates: Analyzing Problems in Context* – Student Handout: *Analyzing Related Rates Problems* – Student Handout: *The Sucker Project* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 68-70)

<table>
<tr><th>Instructional Emphasis for Focus Skills (IPR Skills)</th><th>Resources</th></tr>
<tr><td>Analyzing Problems in Context
Analyzing problems involving rates of change in applied contexts
Students need to be able to extract information from word problems that involve rates of change in contextual situations. Instruction should emphasize identifying key words, evaluating the derivative function to find an instantaneous rate of change, and determining appropriate units for that situation.</td><td>• Focus Topic Lesson Plan: Analyzing Problems Involving Rates of Change in Applied Contexts</td></tr>
<tr><td>Analyzing Problems in Context
Analyzing problems involving related rates
Students need to be able to recognize the key words in a problem that indicate it involves related rates, and determine an appropriate equation to relate the variables in that situation. Instruction should emphasize using strategies such as marking the text and drawing diagrams to translate the contextual situation into a mathematical representation.</td><td>• Instructional Strategy: Marking the Text (CED p. 35)
• Teacher Module: Related Rates: Analyzing Problems in Context
  – Video: Teaching the Skill of Analyzing Problems in Context with Challenging Topics: Related Rates
  – Activity Description: Analyzing Related Rates Problems
• Focus Topic Lesson Plan: Analyzing Problems Involving Related Rates</td></tr>
</table>

<table>
<tr><th>Instructional Emphasis for Learning Objectives</th><th>Resources</th></tr>
<tr><td>LO 2.3D Solve problems involving rates of change in applied contexts
Students should be able to determine information about a rate of change in a particular situation, using the function’s derivative.</td><td>• Concept Outline (CED p. 16)
• Instructional Strategy: Create a plan (CED p. 33)</td></tr>
<tr><td>LO 2.3C Solve problems involving related rates, optimization, rectilinear motion, (BC) and planar motion
Students should be able to translate a problem involving related rates or optimization into a mathematical representation of that situation. Instruction should emphasize how to set up the problem – i.e., determining which quantities are rates and which are variables, identifying the “freeze frame” moment, and determining an appropriate relating equation.</td><td>• Concept Outline (CED p. 15)
• APSI Lesson 18: Curricular Requirements and Syllabus Development (p. 200, 206)</td></tr>
</table>

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE — **Focus Topic** — **Focus Quiz Available** — **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1 (Focus Topic; Focus Quiz Available; Lesson Plan Available)	Analyzing Problems Involving Rates of Change in Applied Contexts	2.3D	• The derivative can be used to express information about rates of change in applied contexts (EK 2.3D1).	While derivatives are easily represented as the slope of a tangent line, students sometimes have trouble seeing derivatives as rates of change in other contexts. Here students are introduced to contextual problems where students need to recognize situations where derivatives are necessary, and then solve those contextual derivative problems. (See for example the "snow problem": AP Calculus AB Exam 2010 FRQ #1)	Analyzing Problems in Context
2	An Introduction to Related Rates (Using Implicit Differentiation)	2.1C 2.3C	• The chain rule is the basis for implicit differentiation (EK 2.1C5). • The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known (EK 2.3C2).	Up until this point, students have mainly worked with equations in terms of x and y. Related rates problems introduce a wide variety of variables, both in terms of quantities that are changing, as well as they variable they are changing with respect to. Students need to practice taking implicit derivatives with respect to t, as well as other variables. Given a problem stem and part of the problem already set up, this lesson should show students how to go through the steps for solving.	Selecting and Applying Procedures
3 (Focus Topic; Focus Quiz Available; Lesson Plan Available)	Analyzing Problems Involving Related Rates	2.3C	• The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known (EK 2.3C2).	Related rates problems are challenging for students, who try to memorize different “types” of problems rather than looking for a common underlying structure. This lesson should focus not on the solving of problems, but rather, on looking at the problem stem and translating the mathematical expressions that are necessary to solve the problem. In this way, students will be better prepared for problem solving in general and less inclined to get stuck when the relevant equations or problem set-ups change.	Analyzing Problems in Context

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Introduction to Optimization	2.3C	• The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval (EK 2.3C3).	What does it mean to optimize something? Students will need to figure out the equation with the variable to be maximized or minimized and recognize that a derivative (often implicit) is called for. They also need to use another equation and substitute into the first equation. This means a higher level of problem-solving. As with the previous lesson, the focus of instruction should be on patterns rather than trying to categorize certain "types" of optimization problems.	Analyzing Problems in Context
5	Optimization	2.3C	• The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval (EK 2.3C3).	Once students understand how to approach optimization problems, they need experience working with different scenarios. Maximizing volume given materials and minimizing distances are classic examples, and they also require students to clearly show that the values they calculate are correct (e.g. verify that the value is a maximum).	Analyzing Problems in Context
6	Related Rates, Optimization, and Interpreting the Meaning of a Derivative in Context	2.3A 2.3C	• The unit for $f'(x)$ is the unit for f divided by the unit for x (EK 2.3A1). • The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known (EK 2.3C2). • The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval (EK 2.3C3).	The purpose of this lesson is to look at a combination of related rates and optimization problems to give students an opportunity to practice and focus on the meaning of the derivatives within the problems. Students at this point have mechanical fluency in taking implicit derivatives, but sometimes have trouble connecting those derivatives to the meaning in a problem. For example, students might be able to take a derivative for the changing area of a circle, but not recognize what dr/dt means. Or, when solving for something like dr/dt, not be able to say what dr/dt means in the context of the problem.	Analyzing Problems in Context

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
7	Connecting Position, Velocity, and Acceleration; Speed versus Velocity	2.3A 2.3C	• The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable (EK 2.3A2). • The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration (EK 2.3C1).	Motion problems are one of the contexts where students are most easily able to "see" what a derivative means. Students can look at graphs and calculate appropriate functions and values on these functions. Students also need to understand the difference between speed and velocity. The AP exam often asks "particle motion" problems, and students who do not understand that velocity has direction have trouble interpreting these problems. Should also emphasize the difference between average velocity vs. instantaneous velocity; same with acceleration.	Analyzing Problems in Context
8	Analyzing Motion Problems	2.3A 2.3C	• The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable (EK 2.3A2). • The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration (EK 2.3C1).	Students will work with contextual motion problems -- analytic, graphical, and tabular – in multiple ways. For example, a student could be given a position function for particle motion and asked if, at a certain time, the particle is speeding up or slowing down. The students need to look at both the velocity and acceleration at that point, and make a determination given those values.	Analyzing Problems in Context

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 5 Test
- ☐ Target the **scoring focus** for Unit 5 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 5 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Analyzing Problems in Context*

Practice Question Description	Scoring Focus for Unit 5
Analyzing Problems in Context Free-Response Question Part (a): Students must take a given equation (with variables *r*, *h*, and *V*) and take a derivative with respect to *t*. Part (b): Using the implicit derivative found in part (a), students must solve for one of the rate variables, given certain conditions, and explain the meaning of their answer in the context of the problem. Part (c): The introduction of a new criterion changes the equation slightly, so students must differentiate and solve a new equation in context.	Part (a): This part of the problem focuses on the mechanics of taking an implicit derivative. Part (b): Students must use the context of the problem to find a relationship between $\frac{dr}{dt}$ and $\frac{dh}{dt}$, which is then used to solve for $\frac{dh}{dt}$. Part (c): Letting *r = 0* means that $\frac{dr}{dt}=0$, and students must make this connection in the process of solving for $\frac{dh}{dt}$.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 5
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students can analyze problems involving optimization and using the First Derivative Test to determine maximum and minimum values. Students should also be able to: • Identify key and relevant information provided and needed to solve problems • Extract and interpret mathematical content from information presented graphically, numerically, analytically, and verbally to solve problems • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Assess the reasonableness of results or solutions • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked

Free-Response Question	Scoring Expectation for Unit 5
FRQ #2 (Calculator)	Focus should be placed on ensuring students can analyze problems involving rates of change in applied contexts. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Apply technology strategically to solve problems • Identify or establish claims, hypotheses, or conjectures • Develop conjectures based on exploration with technology • Accurately report information provided by technology • Connect numerical solutions with mathematical concepts

AP Calculus		Unit 6
Accumulation and Riemann Sums	**Suggested Length:** 7-12 class periods	

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Definite Integrals: Interpreting Notational Expressions*

OVERVIEW/CONTEXT

- In this unit, students are introduced to the concept of area under a curve and the technique of using rectangles to easily approximate the area.
- Students should focus on using the correct notation, interpreting summation notation and connecting it to integral notation, and also connecting the idea of the integral with accumulation.
- Students may struggle with making meaning of summation symbols and remembering what these symbols represent, especially as they relate to Riemann sums and area under a curve. This is especially true when students must connect a Riemann sum with its integral expression.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 5: Building notational fluency *Connect notation to definitions* Students should be able to interpret the notation they are reading, and understand what it represents. Instructional emphasis in this unit should be on precision of language when reading the notation aloud, and repeatedly connecting the symbols in the expression to what they stand for in another representation (e.g., on a graph).	• Teacher Module: *Definite Integrals: Interpreting Notational Expressions* – Student Handout: *The Next Big Idea of AP Calculus: The Definite Integral* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 77-79)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Approximating Values and Functions *Approximating a definite integral using left and right rectangles* Students need to be able to subdivide a particular interval into equal parts, and use those parts to construct rectangles whose heights correspond to the function value either on the left-hand vertex or the right-hand vertex of the rectangle, then use the sum of the areas of those rectangles to approximate the area between the curve and the *x*-axis.	• Instructional Strategy: *Graph and switch* (CED p. 34) • Focus Topic Lesson Plan: *Approximating a Definite Integral Using Left and Right Rectangles*
Interpreting Notational Expressions *Interpreting summation notation and definite integral notation* Students need to be able to read and interpret meaning of out mathematical symbols, particularly when translating between sigma notation and integral notation (and vice versa). Instruction should emphasize how to decode mathematical notation, and students should be given guided practice to help them internalize the meaning of that notation for different types of problems.	• Instructional Strategy: *Notation Read Aloud* (CED p. 36) • Teacher Module: *Definite Integrals: Interpreting Notational Expressions* – Video: *Teaching the Skill of Interpreting Notational Expressions with Challenging Topics: Definite Integrals* – Activity Description: *Introducing the Definite Integral* • Focus Topic Lesson Plan: *Interpreting Definite Integral Notation as the Limit of a Riemann Sum* • Focus Topic Lesson Plan: *Interpreting Summation Notation*

Instructional Emphasis for Learning Objectives	Resources
LO 3.2B Approximate a definite integral Students should be able to find an approximation for a definite integral using Riemann sums; the instructional emphasis here is on left and right Riemann sums.	• Instructional Strategy: *Look for a pattern* (CED p. 35) • Concept Outline (CED p. 17)
LO 3.2A(b) Express the limit of a Riemann sum in integral notation When given a notational expression for the limit of a Riemann sum, students should be able to write an equivalent expression as a definite integral.	• Concept Outline (CED p. 17)
LO 3.2A(a) Interpret the definite integral as the limit of a Riemann sum When given a definite integral expression representing an accumulation (e.g., of area under a curve), students should be able to write an equivalent expression in Riemann sum notation.	• Concept Outline (CED p. 17) • APSI Lesson 9: *Scaffolding the Mathematical Practices* (p. 102)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE **Focus Topic** **Focus Quiz Available** **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1	Approximating the Area Under a Curve with Rectangles	3.2B	• Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally (EK 3.2B1). • Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or non-uniform partitions (EK 3.2B2).	Students are introduced to the concept of area under a curve and the technique of using rectangles to easily approximate the area. The focus here is not on notation, but rather conceptual understanding of the precursors of Riemann sums.	Approximating Values and Functions
2	Approximating a Definite Integral Using Left and Right Rectangles	3.2B	• Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally (EK 3.2B1). • Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or non-uniform partitions (EK 3.2B2).	In this lesson more specific terminology and processes are introduced. Students should focus on drawing and using left and right rectangles, which is something students often confuse later in the context of Riemann sums.	Approximating Values and Functions
3	Approximating a Definite Integral Using Midpoint and Trapezoidal Sums	3.2B 3.2C	• Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally (EK 3.2B1). • Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or non-uniform partitions (EK 3.2B2). • In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area (EK 3.2C1).	While midpoint and trapezoidal sums do introduce a little more formula work into these calculations, students should still be focusing on the ideas of sums. Focus here should be on both uniform and non-uniform partitions, and trapezoidal sums rather than the Trapezoidal Rule, which is not part of the AP Calculus curriculum.	Approximating Values and Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Interpreting Summation Notation	3.2A(b)	• A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition (EK 3.2A1).	This lesson formally introduces the concept of the Riemann sum. The partitioning in earlier lessons (particularly the left rectangular sums) comes into play here as students associate the notation with integrands and limits. Focus should be on translating the limit of the Riemann sum into integral notation, and showing visually what this notation represents. Unpacking the notation and the meaning of the notation is critical to student understanding.	Interpreting Notational Expressions
5	Interpreting Definite Integral Notation (as the Limit of a Riemann Sum)	3.2A(a)	• The definite integral of a continuous function f over the interval $[a,b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ where x_i^* is a value in the *i*th subinterval, Δx_i is the width of the *i*th subinterval, n is the number of subintervals, and $\max \Delta x_i$ is the width of the largest subinterval. Another form of the definition is $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$, where $\Delta x_i = \frac{b-a}{n}$ and x_i^* is a value in the *i*th subinterval (EK 3.2A2). • The information in a definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral (EK 3.2A3).	Following the pattern of the previous lesson, here students learn how to translate from a definite integral to the notation for a Riemann sum. While it may seem like this lesson is simply the previous one in reverse, it is in fact different and challenging. There are an infinite number of ways to rewrite an integral as the limit of a Riemann sum, so students need to be comfortable with the algebraic concept of translations.	Interpreting Notational Expressions
6	Properties of Sums (Definite Integrals)	3.2C	• Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals (EK 3.2C2).	Exact sums can be found for areas defined by basic geometric shapes (triangles, trapezoids, etc.) by using formulas that allow for exact calculation (note, however, that students do not need to know these formulas for the AP exam).	Selecting and Applying Procedures

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
7	Definite Integral Calculations	3.2B 3.2C	• Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally (EK 3.2B1). • Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or non-uniform partitions (EK 3.2B2). • In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area (EK 3.2C1).	Students can calculate definite integrals using formulas and graphs, including graphs with quarter-circles and semi-circles. Students also begin to apply the idea of limits to Riemann sums to estimate the area and predict what they think the exact area will be. Here students should also be introduced to basic properties of integrals, such as $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ as well as linear combinations of definite integrals.	Selecting and Applying Procedures

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 6 Test
- ☐ Target the **scoring focus** for Unit 6 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 6 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Definite Integrals: Interpreting Notational Expressions*

Practice Question Description	Scoring Focus for Unit 6
Definite Integrals: Interpreting Notational Expressions Free-Response Question Part (a): Students are given the Riemann Sum form of a definite integral and asked to rewrite it using the definite integral notation. Part (b): Students use the properties of a constant times a function and the sum of two functions to find the value of an integral. Part (c): Students use the properties of integrals over adjacent intervals to find the value of an integral. *Note: Part (d) involves content from other units.*	Part (a): Looking at the symbolic representation of a definite integral as the limit of a Riemann sum, students should be able to identify the integrand and the limits of integration. Part (b): Students should be able to apply integration techniques to break a sum into two separate integrals, then apply the properties of constant multiples to calculate the value of the definite integral. Part (c): Students need to use the property that $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ to calculate a definite integral.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 6
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students interpret symbolic expressions, especially those that contain summation notation. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Compute accurate results or solutions, sequencing procedures logically • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked • Connect numerical solutions with mathematical concepts

Free-Response Question	Scoring Expectation for Unit 6
FRQ #2 (Calculator)	Focus should be placed on ensuring students can interpret definite integral notation, and connect it to summation notation, as well as approximating a definite integral using Riemann sums. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Develop conjectures based on exploration with technology • Apply technology strategically to solve problems • Accurately report information provided by technology • Connect numerical solutions with mathematical concepts

AP Calculus		Unit
Antiderivatives and the Fundamental Theorem of Calculus	**Suggested Length:** 16-21 class periods	**7**

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Applying Procedures for Integration by Substitution*

OVERVIEW/CONTEXT

- In this unit, students will learn about the concepts of the antiderivative, the definite integral, and the Fundamental Theorem of Calculus. The FTC is a tool that students will use throughout the remainder of the course.
- Students will also have the opportunity to use technology to understand accumulation, as well as apply procedures for evaluating definite integrals analytically.
- The strategy of using u-substitution tends to be most challenging for students – they may not realize when to apply this method or they may not understand that expression "embedded" within an integral must be accounted for in the process of integrating.
- Students will also learn to apply procedures for separable differential equations to find both generic solutions and specific solutions when given initial conditions.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 3: Implementing algebraic/computational processes *Sequence algebraic/computational procedures logically* When working with problems that involve many steps, students should be able to make logical connections from one step to the next. Teaching students the "big picture" and explicitly addressing the "why" behind each step will help make the meaning of processes clearer.	• Teacher Module: *Applying Procedures for Integration by Substitution* – Student Handout: *Sort-the-Steps for Integration by Substitution* – Student Handout: *Error Analysis for Integration by Substitution* – Activity Description: *Sort-the-Steps and Error Analysis for Integration by Substitution* • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 71-73)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Selecting and Applying Procedures *Applying procedures for u-substitution and separable differential equations* Students need to be able to recognize when substitution and separation of variables are applicable procedures, and they need to be able to complete those procedures fully and correctly. Instruction should explicitly unpack why the technique of substitution is used for certain integrals and not others, why certain differential equations can be separated, and how each step in these processes logically follows the preceding step.	• Instructional Strategy: *Sort the Steps* (CED p. 36) • Instructional Strategy: *Error Analysis* (CED p. 34) • Teacher Module: *Applying Procedures for Integration by Substitution* – Video: *Teaching the Skill of Applying Procedures with Challenging Topics: Integration by Substitution* • Focus Topic Lesson Plan: *U-Substitution: Applying Procedures for Integrals* • Focus Topic Lesson Plan: *Applying Procedures for Separable Differential Equations and General Solutions*

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Justifying Properties and Behaviors of Functions *Interpreting behavior of* $f(x)$ *from* $g = f'(x)$ When given the graph of f, students need to be able to justify relative extrema, points of inflection, increasing/decreasing behavior, and concavity for the graph of $g = f'(x)$.	• Instructional Strategy: *Critique reasoning* (CED p. 34) • Focus Topic Lesson Plan: *Justifying Behavior of* $f(x)$ *from a graph of* $g = f'(x)$

Instructional Emphasis for Learning Objectives	Resources
LO 3.3B(a) Calculate antiderivatives **LO 3.3B(b) Evaluate definite integrals** Students should be able to use the technique of substitution of variables to find an antiderivative of an algebraic function or evaluate a definite integral.	• Concept Outline (CED p. 19) • Instructional Strategy: *Discussion Groups* (CED p. 34)
LO 3.3A Analyze functions defined by an integral When given a function defined by an integral, students should be able to determine behaviors and properties of that new function (e.g., where the function is increasing/decreasing or concave up/down).	• Concept Outline (CED p. 18)
LO 3.5A Analyze differential equations to obtain general and specific solutions Students should be able to recognize when a differential equation is separable; for separable differential equations, students should be able to separate the variables and find a general solution.	• Concept Outline (CED p. 20) • APSI Lesson 8: *Unpacking the Mathematical Practices* (p. 95)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE

Focus Topic **Focus Quiz Available** **Lesson Plan Available**

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1	Basic Antiderivatives	3.1A	• An antiderivative of a function f is a function g whose derivative is f (EK 3.1A1). • Differentiation rules provide the foundation for finding antiderivatives (EK 3.1A2).	Here we introduce the concept of the antiderivative, not yet linking it to previous work. Students know, for example, that the derivative of $y = x^2$ is $y' = 2x$. Reversing the process is not a perfect inverse -- a function whose derivative is $2x$ is of the form $x^2 + C$, where C is any real number. Students should be able to recognize the antiderivatives of basic functions, including $y = \sin(x)$, $y = \cos(x)$, $y = e^x$, and $y = \frac{1}{x}$. Here also we introduce the notation for the antiderivative.	Basic Antiderivatives
2	FTC Part I	3.3B(a) 3.3B(b)	• If f is continuous on the interval $[a,b]$ and F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$ (EK 3.3B2). • The notation $\int f(x)dx = F(x) + C$ means that $F'(x) = f(x)$, and $\int f(x)dx$ is called an indefinite integral of the function f. (EK 3.3B3)	The first part of the FTC is a tool students will use throughout the remainder of AP Calculus. It connects the idea of the Riemann sum with an antiderivative, and it's one of the most important theorems in calculus. Students learn the form of the FTC (and the term definite integrals) using generic notation (f and F) and follow the mechanics of the process.	FTC Part I
3	FTC Part II	3.3A	• If f is a continuous function on the interval $[a,b]$, then $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, where x is between a and b (EK 3.3A2). • Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x) = \int_a^x f(t)dt$. (EK 3.3A3)	The second form of the FTC involves a definite integral where at least one of the limits of integration is a variable. Here we introduce the idea of a dummy variable and link differentiation and antidifferentiation. Special attention should be paid to correct substitution of the limits of integration, as well as how the chain rule plays out due to the limits of integration (for example, if one of the limits is itself a function).	FTC Part II

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Evaluating Definite Integrals (Basic Only)	3.3B(b)	• If f is continuous on the interval $[a,b]$ and F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$. (EK 3.3B2)	Here students practice the idea of antiderivatives in the context of evaluating definite integrals. Students should practice with polynomials that combine basic rules (e.g. $2x+3$) as well as trig functions and other functions, leading to the general rule for the integral of x^n.	Evaluating Definite Integrals (Basic Only)
5	Using Technology to Understand Accumulation	3.3A	• The definite integral can be used to define new functions; for example, $f(x) = \int_0^x e^{-t^2} dt$. (EK 3.3A1)	Students use tables and graphs to analyze the integral from 0 to x of $f(t)dt$. Students are given a specific function $f(t)$ (defined in the first quadrant) and then plug in different values of x, observing the accumulating values. Students use these values to sketch a graph of $F(x)$.	Using Technology to Understand Accumulation
6	U-Substitution: Applying Procedures for Integrals	3.3B(a) 3.3B(b)	• The function defined by $F(x) = \int_a^x f(t)dt$ is an antiderivative of f. (EK 3.3B1) • The notation $\int f(x)dx = F(x) + C$ means that $F'(x) = f(x)$, and $\int f(x)dx$ is called an indefinite integral of the function f (EK 3.3B3). • Many functions do not have closed form antiderivatives (EK 3.3B4).	Strong connection needs to be made to the lesson on basic antiderivatives -- we can calculate antiderivatives for x^n, $\sin(x)$, $\cos(x)$, e^x, $\frac{1}{x}$, and so on. But what happens when things get more complex, e.g. $(2x+3)^n$, $\sin(2x)$, $\cos(0.7x)$, e^{3x}, or $\frac{1}{3x+5}$? Substitution takes something more complicated and reduces it to one of the "basic" forms. Care must be taken to substitute back at the end of the process. In the case of definite integrals, either the function must be returned to the original variable, or the limits of integration must be changed to match the new variable. In this lesson, we focus on basic expressions only.	U-Substitution: Applying Procedures for Integrals

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
7	Integration by Substitution Continued: More Practice and Trigonometric Functions	3.3B(a) 3.3B(b)	• The notation $\int f(x)dx = F(x)+C$ means that $F'(x)=f(x)$, and $\int f(x)dx$ is called an indefinite integral of the function f (EK 3.3B3). • Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, *(BC)* integration by parts, and nonrepeating linear partial fractions (EK 3.3B5).	Following the previous lesson, this lesson focuses on trigonometric functions only. Students should note the pattern of the u and the du here -- for example, when integrating $y=\sin^5(x)\cdot\cos(x)$, $\sin(x)$ is being raised to the 5th power. We don't know how to integrate $y=\sin^5(x)$, so $u=\sin(x)$ must be our substitution. Not surprisingly, our du is $\cos(x)dx$, and there is only a single appearance of $\cos(x)$ in our integrand. Key note: Students need help understanding that the judicious use of a trigonometric identity could help with the antidifferentiation process.	Integration by Substitution Continued: More Practice and Trigonometric Functions
8	Integration by Substitution Continued: Exponential and Logarithmic Functions	3.3B(a) 3.3B(b)	• The notation $\int f(x)dx = F(x)+C$ means that $F'(x)=f(x)$, and $\int f(x)dx$ is called an indefinite integral of the function f. (EK 3.3B3) • Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, *(BC)* integration by parts, and nonrepeating linear partial fractions. (EK 3.3B5)	Following the previous lessons on substitution, this lesson focuses on exponential and logarithmic functions. Though these functions are of course different from trigonometric functions, special attention should be paid to the fact that the process and the idea are exactly the same. Key note: Students often make an error in deciding what to let u equal in an antiderivative.	Integration by Substitution Continued: Exponential and Logarithmic Functions
9	Average Value and Non-Continuous Functions	3.2C 3.4B	• The definition of the definite integral may be extended to functions with removable or jump discontinuities. (EK 3.2C3) • The average value of a function f over an interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$. (EK 3.4B1)	Students are introduced to the average value formula and connect its meaning via use of graphs (showing the area under the curve for a function and the area under the rectangle for the average value, and how they are equal).	Average Value and Non-Continuous Functions

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
10	Interpreting Behavior of $f(x)$ from a Graph of $f'(x)$	3.3A 2.2A	• The definite integral can be used to define new functions; for example, $f(x)=\int_0^x e^{-t^2}\,dt$ (EK 3.3A1). • Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x)=\int_a^x f(t)\,dt$. (EK 3.3A3) • First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection. (EK 2.2A1)	This section focuses on a graphical approach of connecting the graph of a derivative to the function itself and extends similar work done in other lessons in previous units. Students can now work with functions defined by an integral, i.e. functions of the form $y=\int_a^x f(t)\,dt$ where $f(t)$ is a given graph of a first derivative.	Interpreting Behavior of $f(x)$ from a Graph of $f'(x)$
11	Introduction to Differential Equations	3.5A	• Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, *(BC)* and logistic growth. (EK 3.5A1)	Students have produced many differential equations in their work up to this point, but now we focus on the differential equation itself (often in the form of dy/dx = or y' =). Differential equations of the form dy/dx give us information about how the graph of y behaves. Here we begin by looking at equations of the basic form in context (e.g. rate of change of a bacteria population). This idea should then be extended to use variables other than x and y so that students develop notational fluency.	Introduction to Differential Equations
12	Verifying Solutions to Differential Equations	2.3E	• Solutions to differential equations are functions or families of functions. (EK 2.3E1) • Derivatives can be used to verify that a function is a solution to a given differential equation. (EK 2.3E2)	Solving differential equations in general is very complicated -- there are entire college courses just on this idea! Here we will look at what the "solution to a differential equation" means. Students are used to solutions being numbers, not equations, so this is a challenging concept for them. Here students will be given a differential equation and asked to show that a given function is a solution. Alternatively, they can be given a differential equation and multiple possible solutions and asked to verify which, if any, actually are solutions.	Verifying Solutions to Differential Equations

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
13	Applying Procedures for Separable Differential Equations and General Solutions	3.5A	• Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, *(BC)* and logistic growth. (EK 3.5A1) • Some differential equations can be solved by separation of variables. (EK 3.5A2)	As mentioned in the previous lesson, finding solutions to differential equations is challenging. In AP Calculus, the only type of differential equation that students will need to solve is the most basic type, a separable differential equation. Focus here will be on the general solution, focusing somewhat on including the +C at the right time. Emphasis is on reasoning about the steps in the procedure and why they go in the order they do.	Applying Procedures for Separable Differential Equations and General Solutions
14	Differential Equations: Specific Solutions (Initial Conditions)	3.5A	• Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, *(BC)* and logistic growth. (EK 3.5A1) • Solutions to differential equations may be subject to domain restrictions. (EK 3.5A3) • The function F defined by $F(x) = c + \int_a^x f(t)dt$ is a general solution to the differential equation $\frac{dy}{dx} = f(x)$, and $F(x) = y_0 + \int_a^x f(t)dt$ is a particular solution to the differential equation $\frac{dy}{dx} = f(x)$ satisfying $F(a) = y_0$. (EK 3.5A4)	Continuing from the previous lesson, we now introduce the idea of a specific solution. Emphasize here as well is that the proper introduction of +C is essential.	Differential Equations: Specific Solutions (Initial Conditions)
15	Estimating Solutions to Differential Equations: Slope Field	2.3E 2.3F	• Solutions to differential equations are functions or families of functions. (EK 2.3E1) • Derivatives can be used to verify that a function is a solution to a given differential equation. (EK 2.3E2) • Slope fields provide visual clues to the behavior of solutions to first order differential equations. (EK 2.3F1)	Slope fields provide information about a function, even when you don't know the equation of the function (and cannot find it). Students will look at slope fields to draw functions, focusing on how different initial conditions can provide similarly shaped answers, showing that the general solution to a differential equation is a family of functions. Students will also be given a differential equation and asked to draw a slope field, focusing on a small area of interest.	Estimating Solutions to Differential Equations: Slope Field

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
16	Solving Differential Equations in Context: Working with Exponential Growth and Decay	3.5B	• The model for exponential growth and decay that arises from the statement "The rate of change of a quantity is proportional to the size of the quantity" is $\frac{dy}{dt} = ky$. (EK 3.5B1)	Students previously saw limited context for differential equations, but now they will work with them in context. Exponential growth and decay problems allow students to start with equations in the form of dy/dt = ky, where different parts of the equation and the solution have easy-to-understand practical interpretations.	Solving Differential Equations in Context: Working with Exponential Growth and Decay

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 7 Test
- ☐ Target the **scoring focus** for Unit 7 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 7 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Applying Procedures for Integration by Substitution*

Practice Question Description	Scoring Focus for Unit 7
Applying Procedures for Integration by Substitution Free-Response Question *Note: Part (a) involves content from previous units.* Part (b): Students need to find the average value of a function presented graphically. Part (c): Students need to calculate a definite integral where part of the integral is presented graphically, and another part is presented analytically and requires substitution. Part (d): Students need to calculate a definite integral for a function presented graphically, and the integral requires substitution.	Part (b): Students earn points for setting up and for solving for the average value of the function. Part (c): Students earn points here for correct integration for the graphical portion of the problem, as well as for the antidifferentiation and evaluation of the analytical integral. Part (d): Students earn points here for supporting work – setting up the substitution for the integrand, and setting up the new limits of integration – as well as for the final answer.

UNIT TEST

For each unit in AP Calculus, there is a full, pre-assembled, Unit Test, which you may choose to assign to your students. This Unit Test will concentrate on the emphasized unit focus practices and skills in the context of the unit content. This assessment includes 11 multiple-choice questions and 2 free-response questions.

The free-response questions throughout the course have been designed to build students' skills over the course of the year. For this reason, the scoring expectations and rubrics have been modified to reflect the instructional emphasis of the unit. Use these modified rubrics provided in Academic Merit as you assess students' progress toward mastery.

Note: The purpose of the Unit Test is not to predict how your students will do on the summative AP Exam; rather, it is meant to determine the level to which your students have mastered the focus practices and skills of the unit.

Free-Response Question	Scoring Expectation for Unit 7
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students can apply procedures for evaluating integrals, focusing on u-substitution, as well as justifying the behavior of a function from the graph of its derivative. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Compute accurate results or solutions, sequencing procedures logically • Apply definitions and theorems to provide evidence for supporting claims or conjectures, or justifying conclusions or solutions • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked • Connect numerical solutions with mathematical concepts

Free-Response Question	Scoring Expectation for Unit 7
FRQ #2 (Non-calculator)	Focus should be placed on ensuring students can apply procedures for solving separable differential equations and finding both general and specific solutions, as well as constructing slope fields. Students should also be able to: • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Apply technology strategically to solve problems • Compute accurate results or solutions, sequencing procedures logically • Develop conjectures based on exploration with technology • Accurately report information provided by technology

AP Calculus		Unit 8
Applications of Definite Integrals	**Suggested Length:** 11-16 class periods	

PLAN

- ☐ Review the **Curriculum Framework** in the AP Calculus Course and Exam Description (CED)
- ☐ Identify **Instructional Approaches** section in the CED that will be useful in teaching the **focus skills**
- ☐ Watch the **Teaching Module** *Interpreting Context for Definite Integrals*

OVERVIEW/CONTEXT

- In this unit, students will look at accumulation problems in context. Special focus should be placed on being able to translate a contextual scenario into integral notation, paying attention to units of measure. In order to do this, students first need to identify that a problem involves accumulation and identify and extract important, relevant information needed to solve the problem.
- Students will also use definite integrals to solve problems involving motion, areas under and between curves, and volumes of solids formed when regions are rotated around horizontal and vertical lines or have known cross-sections.
- Just like with derivatives, application problems with integrals often prove challenging for students – they may choose the wrong process and differentiate when they should integrate, or they may not understand how to connect a given rate function with the function that represents a quantity.

INSTRUCTIONAL FOCUS FOR THE UNIT

Instructional Emphasis for Mathematical Practices (MPACs)	Resources
MPAC 6: Communicating *Explain the meaning of expressions, notation, and results in terms of a context (including units)* Students should be able to express their results with a verbal statement that includes a noun, a verb, and appropriate units for the given situation. Providing templates and targeted feedback will help students improve when writing these types of interpretations.	• APSI Lesson 6: *Understanding the Mathematical Practices* (p. 80-82)
MPAC 2: Connecting concepts *Identify a common underlying structure in problems involving different contextual situations* Students should be able to recognize the commonalities across problems that involve motion or accumulation.	• Instructional Strategy: *Marking the Text* (CED p. 35) • APSI Lesson 6: *Understanding the Mathematical Practices* (p. 68-70)

Instructional Emphasis for Focus Skills (IPR Skills)	Resources

Interpreting Context
Accumulation and net change in context
Students need to be able to explain the meaning of a definite integral in context. Instruction should emphasize using an appropriate noun, verb, and units in their responses.
	• *Communicating in Mathematics* (CED p. 37-38)
	• Teacher Module: *Interpreting Context for Definite Integrals*
	  – Video: *Teaching the Skill of Interpreting Context with Challenging Topics: Definite Integrals*
	  – Student Handout: *Interpreting Definite Integrals*
	  – Activity Description: *Using Sentence Diagramming to Interpret Definite Integrals*
	• Focus Topic Lesson Plan: *Accumulation and Net Change in Context*

Instructional Emphasis for Focus Skills (IPR Skills)	Resources
Analyzing Problems in Context *Analyzing problems involving definite integrals and accumulation or motion* Students need to be able to translate a given situation in which something is accumulating or in motion into a mathematical expression involving an integral. Instruction should emphasize identifying key words, and translating that vocabulary into expressions that represent distance, speed, velocity, and acceleration.	• Focus Topic Lesson Plan: *Analyzing Problems Involving Definite Integrals and Accumulation* • Focus Topic Lesson Plan: *Analyzing Problems Involving Definite Integrals and Motion*

Instructional Emphasis for Learning Objectives	Resources
LO 3.4A Interpret the meaning of a definite integral within a problem Students should be able to interpret a definite integral as an accumulation of a particular thing in a given situation.	• Concept Outline (CED p. 19)
LO 3.4E Use the definite integral to solve problems in various contexts Students should be able to identify when the use of a definite integral is appropriate for a given situation (i.e., when the problem involves accumulation), and extract the information necessary to answer the question being asked.	• Concept Outline (CED p. 20)
LO 3.4C Apply definite integrals to problems involving motion Students should be able to determine whether the problem involves speed or velocity, when to use the initial condition to find a new position, and when to not use the initial condition (i.e., when finding displacement).	• Concept Outline (CED p. 19)

TEACH

- ☐ Teach each lesson. Emphasize the **highlighted focus topics**, supplementing them with teacher-designed practice activities for the foundational concepts
- ☐ Use provided **lesson plans** (indicated with icons below) to reinforce skill development and conceptual understanding
- ☐ Remember that **focus topics are critical**, and that students should master these skills and content
- ☐ Assign **Focus Quizzes** to check for understanding

INSTRUCTIONAL SEQUENCE

Focus Topic | Focus Quiz Available | Lesson Plan Available

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
1 (Focus Topic; Focus Quiz Available; Lesson Plan Available)	Accumulation and Net Change in Context	3.4A 3.4E	• A function defined as an integral represents an accumulation of a rate of change (EK 3.4A1). • The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval (EK 3.4A2). • The definite integral can be used to express information about accumulation and net change in many applied contexts (EK 3.4E1).	The concepts of initial condition and accumulation have been alluded to in the FTC part I, but in this lesson we make the process explicit and concrete. The integral from a to b represents an accumulation of something; $F(a)$ is the starting amount or initial condition, and $F(b)$ is the ending amount. Students will look at accumulation problems in context and focus on identifying these parts and using sentences to discuss them in context using correct units.	Interpreting Context
2 (Focus Topic; Focus Quiz Available; Lesson Plan Available)	Analyzing Problems Involving Definite Integrals and Accumulation	3.4E	• The definite integral can be used to express information about accumulation and net change in many applied contexts (EK 3.4E1).	This lesson focuses on the problem-solving required for working with problems in context. Students need to identify that a problem involves accumulation (for example, rate-in and rate-out problems) and extract important information to answer given questions. Students will use what they learned in the lesson about accumulation in context to solve problems that ask for the change in amount of something versus the final amount of something, which means focusing on whether the initial condition is needed.	Analyzing Problems in Context
3	Integration and Motion Problems	3.4C	• For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time (EK 3.4C1).	Students have worked with motion problems in a previous Unit when calculating derivatives, but now they'll be working in the opposite direction (i.e. acceleration to velocity to position). Here students should see problems involving particles in rectilinear motion over an interval of time, and understand that the definite integral of velocity represents the particle's displacement over the interval, and that the definite integral of speed represents the particle's total distance traveled over the interval of time.	Analyzing Problems in Context

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
4	Analyzing Problems Involving Definite Integrals and Motion	3.4C	• For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time (EK 3.4C1).	A lot of motion problems look and sound the same, so the focus in this topic is on analyzing the problem stem for keywords that indicate how to set up the problem. That is, when presented with a motion problem, identifying key components like initial conditions, focusing on key words like "displacement" or "distance traveled," and determining whether an integral or a derivative is appropriate.	Analyzing Problems in Context
5	Area: Vertical Rectangles and Areas Between Curves	3.4A 3.4D	• The limit of an approximating Riemann sum can be interpreted as a definite integral (EK 3.4A3). • Areas of certain regions in the plane can be calculated with definite integrals. *(BC)* Areas bounded by polar curves can be calculated with definite integrals (EK 3.4D1).	Students have been exposed to the idea of area and definite integrals, and this idea is extended to areas between curves (calculated vertically). Students will go through the Riemann sum process, as before, to see that the height of the rectangles is (top function - bottom function). Examples where one or both curves is in the 4th quadrant shows that location on the Cartesian coordinate system is irrelevant, which is a common student misconception.	Analyzing Problems in Context
6	Area: Horizontal Rectangles and Areas Between Curves	3.4D	• Areas of certain regions in the plane can be calculated with definite integrals. *(BC)* Areas bounded by polar curves can be calculated with definite integrals (EK 3.4D1).	Despite the fact that the process for calculating areas between curves with horizontal rectangles is almost identical to the method for vertical rectangles, students struggle with this concept. Examples should be given of equations of the x = form rather than the y= form, and students will draw horizontal rectangles to follow a similar process to what they have previously done.	Analyzing Problems in Context
7	Volumes: Disc Method	3.4D	• Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals (EK 3.4D2).	By now students have seen the Riemann sum applied numerous times, but always with the area of a rectangle – something like $f(x)\cdot\Delta x$. This idea was extended to the integral, with the Δx relating to dx. Now students take a region in the first quadrant, flush with the x-axis, and rotate it around the x-axis. Students will draw rectangles to simulate discs, each of which has a volume of $\pi\cdot\left[f(x)\right]^2\cdot\Delta x$. Students will relate this to previous Riemann sums to establish the formula for volumes of revolution with the disc method.	Analyzing Problems in Context

	Topic	Learning Objective	Content Emphasis	Instructional Focus	Skill Emphasis
8	Volumes: Washer Method	3.4D	• Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals (EK 3.4D2).	The previous lesson will be extended to include two functions that are not flush with the x-axis being rotated around the x-axis. Given $f(x)$ and $g(x)$, rotate $f(x)$ around the x-axis as before to find the volume. Do the same for $g(x)$. Then ask what happens when the region enclosed by $f(x)$ and $g(x)$ is rotated around the x-axis. Visuals help show that the resulting volume is simply the difference between the two values previously calculated, and that the "washer method" is really just doing the disc method twice and subtracting. Key takeaway: clearly identifying which function is the "top" function and which is the "bottom" function will prevent many simple mistakes.	Analyzing Problems in Context
9	Volumes: Washer Method about Various Horizontal Lines	3.4D	• Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals (EK 3.4D2).	This lesson extends the previous lesson on the washer method to rotate around other lines. Relative position of the functions and the axis make it very important for students to label the "top" and "bottom" functions to obtain a correct answer.	Analyzing Problems in Context
10	Volumes: Washer Method about Vertical Lines	3.4D	• Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals (EK 3.4D2).	Following the lesson on horizontal rectangles and rotating around various horizontal axes, students will use the disc method around the y-axis and the washer method around the y-axis and other vertical lines.	Analyzing Problems in Context
11	Volumes with Known Cross-Sections: Right Triangles, Squares, and Semi-Circles	3.4D	• Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals (EK 3.4D2).	With the disc method, we're creating an object that has cross sections that are circles. The idea of solids with known cross sections extends this idea. For the disc method, the formula we derived was the integral for the area of a circle. When the cross sections have different shapes, the integrand is simply the new area formula. Cross sections with areas of squares, semi-circles, and right triangles should be used.	Analyzing Problems in Context

ASSESS

- ☐ Assign the **Teaching Module Practice Question** and give feedback to students before they take the Unit 8 Test
- ☐ Target the **scoring focus** for Unit 8 as you apply the rubric to student work
- ☐ Use **Focus Quizzes** to assess student mastery of foundational and important course content and skills
- ☐ Create additional practice opportunities using the **AP Question Bank** and assign them through Academic Merit

PRACTICE QUESTION

Before taking the Unit 8 Test, students may benefit from additional opportunities to receive feedback. You may assign the following sample free-response question from the Teaching Module to students and score your students' work according to the question rubric.

Module – *Interpreting Context for Definite Integrals*

Practice Question Description	Scoring Focus for Unit 8
Interpreting Context for Definite Integrals Free-Response Question Part (a): In a rate in/rate out problem, students need to determine whether the amount of something is increasing or decreasing at a specific time. Part (b): Students need to calculate the amount of something that has accumulated over a time interval. Part (c): Students need to determine how much of something is present at a certain time. *Note: Part (d) involves content from other units*.	Part (a): Students need to compare two rate functions at a specific time to determine whether the amount is increasing or decreasing. Points are earned for analysis and answer. Part (b): Students need to set up a correct integral, and points are earned for set-up and answer. Part (c): Students need to set up an integral and use an initial condition. Points are earned for the correct integrand, use of the initial condition, and correct answer.

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Free-Response Question	Scoring Expectation for Unit 8
FRQ #1 (Non-calculator)	Focus should be placed on ensuring students apply definite integral to problems involving area and volume. Students should also be able to: • Identify key and relevant information provided and needed to solve problems • Extract and interpret mathematical content from information presented graphically, numerically, analytically, and verbally to solve problems • Select appropriate mathematical methods or procedures to solve problems, sequencing procedures logically • Assess the reasonableness of results or solutions • Use accurate mathematical vocabulary and notation • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked

Free-Response Question	**Scoring Expectation for Unit 8**
FRQ #2 (Calculator)	Focus should be placed on ensuring students can apply definite integrals to problems involving motion, accumulation, and net change in context. Students should also be able to: • Apply technology strategically to solve problems • Develop conjectures based on exploration with technology • Present precise conclusions and solutions graphically, numerically, analytically, and verbally—specifying units of measure when necessary—that answer the questions asked • Accurately report information provided by technology • Connect numerical solutions with mathematical concepts