AP® Calculus BC

Course Planning and Pacing Guide

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CollegeBoard

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Welcome to the AP Calculus BC Course Planning and Pacing Guides

This guide is one of several course planning and pacing guides designed for AP® Calculus BC teachers. Each provides an example of how to design instruction for the AP course based on the author's teaching context (e.g., demographics, schedule, school type, setting). These course planning and pacing guides highlight how the components of the *AP Calculus AB and BC Curriculum Framework*, which uses an Understanding by Design approach, are addressed in the course. Each guide also provides valuable suggestions for teaching the course, including the selection of resources, instructional activities, and assessments. The authors have offered insight into the *why* and *how* behind their instructional choices — displayed along the right side of the individual unit plans — to aid in course planning for AP Calculus teachers.

The primary purpose of these comprehensive guides is to model approaches for planning and pacing curriculum throughout the school year. However, they can also help with syllabus development when used in conjunction with the resources created to support the AP Course Audit: the Syllabus Development Guide and the four Annotated Sample Syllabi. These resources include samples of evidence and illustrate a variety of strategies for meeting curricular requirements.

Contents

Instructional Setting

Illinois Mathematics and Science Academy ▸ Aurora, IL

School

Illinois Mathematics and Science Academy (IMSA) is a public, residential school, grades 10–12, for students across the state of Illinois. Admission is by application, looking for interest and talent in math and science. Students come from a wide range of backgrounds and experiences — strong suburban schools, inner-city Chicago, and small farm towns downstate.

Student population

We have about 650 students across three grades. Currently, our student body is approximately 45 percent Asian, 36.5 percent Caucasian, 8.5 percent Latino, 5.5 percent Black, and 4.5 percent biracial. About 12 percent qualify for free or reduced lunch. Typically, all of our students go on to a four-year college or university, though occasionally one or two students each year may choose to defer.

Instructional time

We start classes in late August and finish the first semester before winter break. Exams for the second semester are given right after Memorial Day weekend. Classes meet four days per week and are 55-minutes long. This gives about 65 class meetings per semester. The majority of our students finish the Calculus BC program, and a much smaller number take the AB program. To do this with our approach to teaching and work in class, we intentionally slow the pace and use three semesters for Calculus BC. However, this pacing guide has been significantly modified to be much more representative of a two-semester course.

Student preparation

Students take a placement test when starting at IMSA so that they're put into an appropriate course. In calculus, we have a mixture of 10th to 12th graders.

Primary planning resources

Hughes-Hallett, Deborah, et al. *Calculus: Single Variable*. 6th edition. Hoboken, NJ: Wiley, 2013.

Mathematica. Wolfram Research, Inc. http://www.wolfram.com/mathematica/.

Overview of the Course

Primarily I want my students to think mathematically about the concepts behind calculus, and I want them to talk to one another about their understanding of the ideas and about problems. To encourage this, much of the learning of new material in class is done by way of developmental worksheets. In some cases, such as L'Hospital's Rule, this is simply an introduction to a concept. The goal is to make students see the issue, the need for a new approach, or the basic concept behind the method. In other cases, such as for the introductory study of limits, the sequence of worksheets includes many different situations, notation, and a variety of problems. In either case, these guide students through a sequence of problems and questions. They may include special cases, ask for comparisons between problems, ask students to make observations about graphics, or they may simply progress through problems of increasing difficulty. For some topics, there are short one-pagers. For other topics there may be a sequence of 8 or 10 worksheets, each with multiple pages.

Students sit at tables in groups of three or four and talk to one another. They share different voices, new ideas, and attempt to articulate their understanding of the material. The ability to communicate mathematics is an essential skill, and my students practice this frequently. At times, one will start, pause, and then another will be able to extend the idea. Often they check each other's work. Students are accustomed to speaking with one another, in small groups and in front of the class, in many classes at IMSA. Without question, this allows students to be more comfortable with our class format.

As the instructor, I wander from table to table to offer help. I can give suggestions, ask questions, check answers, and ask about "creative" algebraic steps. Sometimes I'll ask how a problem relates to another one they've already done. Sometimes I'll ask them to make a graph. I can help with calculator or computer work. I may offer another problem as a gentler step. For stronger, faster students, I may ask a harder question, challenging them to think a little more deeply. This allows some flexibility to deal with students of differing abilities in our class.

I collect some worksheets to offer formative feedback on analytic work and on written explanations. Also, and very importantly, with the whole class, I pull ideas together to clarify concepts, notation, hypotheses, or special cases. I reinforce the major ideas and offer more practice on the board for all to see. Still, whenever possible, I believe it is very important for students to be exposed to ideas and to think first, before simply being told how things work.

As a former student said, "Calculus is about change, so we ought to see things changing." With this in mind, I show a lot of computer animations in class. Using technology has helped me to understand calculus more thoroughly, and I believe that my students benefit from tying together the analytical and graphical approaches. Students use calculators and computers frequently, both on their own and with my instruction. I write computer files where they are expected to edit functions or values and see the effects. My students consistently report that they appreciate the visualizations.

Important note: The preceding paragraphs explain our three-semester course. However, the units in the pacing guide that follows have been significantly modified at the request of the College Board to be much more representative of a two-semester course. To accomplish this, there were two major changes. First, I have removed many topics that I normally teach, but are not required by the AP Calculus Curriculum Framework. These include an introduction to the delta-epsilon definition of a limit, economics, shells, and several more techniques of integration. For many more ideas, including L'Hospital's Rule, related rates, techniques of integration, and error bounds for Taylor series approximations, I normally include more depth than is required. Second, and probably more importantly, two semesters allows less time for students to work together during class. This does affect how they connect concepts, learn to communicate ideas, and learn from one another.

Mathematical Thinking Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

MPAC 1: Reasoning with definitions and theorems

Students can:

a. use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;

b. confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;

c. apply definitions and theorems in the process of solving a problem;

d. interpret quantifiers in definitions and theorems (e.g., “for all,” “there exists”);

e. develop conjectures based on exploration with technology; and

f. produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2: Connecting concepts

Students can:

a. relate the concept of a limit to all aspects of calculus;

b. use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

c. connect concepts to their visual representations with and without technology; and

d. identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

Students can:

a. select appropriate mathematical strategies;

b. sequence algebraic/computational procedures logically;

c. complete algebraic/computational processes correctly;

d. apply technology strategically to solve problems;

e. attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and

f. connect the results of algebraic/computational processes to the question asked.

Mathematical Thinking Practices for AP Calculus (MPACs)

MPAC 4: Connecting multiple representations

Students can:

a. associate tables, graphs, and symbolic representations of functions;

b. develop concepts using graphical, symbolical, or numerical representations with and without technology;

c. identify how mathematical characteristics of functions are related in different representations;

d. extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);

e. construct one representational form from another (e.g., a table from a graph or a graph from given information); and

f. consider multiple representations of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

Students can:

a. know and use a variety of notations (e.g., $f'(x), y', \frac{dy}{dx}$);

b. connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);

c. connect notation to different representations (graphical, numerical, analytical, and verbal); and

d. assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

Students can:

a. clearly present methods, reasoning, justifications, and conclusions;

b. use accurate and precise language and notation;

c. explain the meaning of expressions, notation, and results in terms of a context (including units);

d. explain the connections among concepts;

e. critically interpret and accurately report information provided by technology; and

f. analyze, evaluate, and compare the reasoning of others.

Pacing Overview

Unit	Hours of Instruction	Unit Summary
1: Introduction to Rates and Limits	12	Students explore relationships between simple velocity and position graphs. Homework focuses on a review of precalculus topics that will be essential for this course, including basic functions and absolute value as distance. Calculation of limits and concept of continuity are introduced.
2: Meaning of the Derivative	13	We extend our study of measuring velocity and speed, clarifying the importance of $\frac{\Delta y}{\Delta x}$, and we do more graphing, moving back and forth between the function and the derivative function. We examine general shapes of graphs and develop the vocabulary about increasing/decreasing functions and concavity. We begin to find approximations of the derivative at specific points. This leads to a formal definition of a derivative, and we then ask about the meaning of the derivative in a variety of contexts.
3: Calculating Derivatives	22	This unit uses the definition of the derivative to develop rules for computing derivatives of elementary functions. Practice is given for each rule, and combinations of functions are also examined. Along the way, students are asked to use derivatives to solve other problems about tangent lines, rates, and graphs. At the end, we connect average and instantaneous rates of change with the Mean Value Theorem.
4: Applications of Derivatives	15	This unit reviews and extends many ideas introduced earlier that connect to properties of graphs, tangent lines, and optimization. New ideas of related rates, L'Hospital's Rule, and parametrics are studied.
5: Meaning of the Integral and the Fundamental Theorems of Calculus	17	Here, we begin the study of the integral. Connections are made repeatedly among the definition, a geometrical meaning and area, the role of accumulation, and the Fundamental Theorems. Ideas are introduced, studied, and revisited as these concepts are tied together.
6: Techniques of Integration	12	Various methods of integration are studied. An important part of the learning is for students to choose an appropriate method.
7: Applications of Integrals	9	This unit covers the use of integrals for area, volume, and arc length.

Pacing Overview (continued)

Unit	Hours of Instruction	Unit Summary
8: Improper Integrals	5	This short unit on improper integrals creates a bridge from the definite integrals previously studied to integrals with unbounded intervals that lead to infinite series.
9: Sequences and Series	13	In this unit, students study sequences and series, and what it means for each of these to converge. We will study various tests of convergence for infinite series. We will look at alternating series and the alternating series test.
10: Taylor Series	12	This unit studies Taylor series — how they are formed, their graphs, convergence, error bounds, and how to use the Taylor series for one function to create the series for a related function.
11: Polar Graphs and Vectors	8	We do a quick review of basic graphing of polar functions, followed by a study of tangent lines and area. After polar functions, we spend a couple of days on vectors, connecting these back to parametrically defined curves.
12: Differential Equations	10	In this introduction to differential equations (DEs), we first examine what a DE is. We study different approaches to understanding them — slope fields, Euler's method, and analytic solutions. Finally, we use these approaches to study a variety of models.

UNIT 1: INTRODUCTION TO RATES AND LIMITS

BIG IDEA 1
Limits

Essential Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How do position and velocity relate? ▸ How do we evaluate basic limits? ▸ What do limits tell us about the behavior of functions? ▸ What is a continuous function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.		**Instructional Activity: Velocity and Position** Students do worksheets in class showing simple graphs. We start with simple, piecewise-linear graphs of position, and sketch velocity. We then reverse the situation, moving from velocity to possible position graphs. Graphs in both directions gradually become more involved. I begin discussions of $\frac{\Delta s}{\Delta t}$ to introduce slope. With more involved graphs, I begin using vocabulary of local extrema and concavity. For example, with more involved position graphs I ask students to put multiple possible velocity graphs on the board to compare.	*I want students to think about rates of change in a familiar context — motion. I look for rough graphs and basic relationships. (Homework is review of some precalculus topics.)* *This activity and the formative assessment are preliminary work for the learning objective. We will return to this in more depth later.*
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.		**Formative Assessment: Drawing Graphs** In this assessment I ask students to sketch graphs, moving both from position to velocity and from velocity to position.	*I want to see how well students are making connections between position and velocity. Simple graphs and basic relationships are fine at this point. I collect the graphs and give feedback to students.*
LO 2.3F: Estimate solutions to differential equations.	**Software** We use *Mathematica*, though graphing calculators or other software such as *Excel* can be used equally well.	**Instructional Activity: Introduction to Euler's Method** I introduce Euler's method with very simple examples of the form $y' = f(x)$. We also speak about $v = v(t)$ to make tables of ordered pairs and graphs of the position function. For example, I use Euler's method with $y' = 3x - 1$ on the interval $0 \le x \le 6$ with step size 2, beginning with $y(0) = 0$. The effect of a smaller step size is examined with computer files. More good graphs of y and y' are observed on the computer.	*Euler's method:* *$new\ y = old\ y + rate \cdot \Delta x$ is a basic concept for tangent lines and other concepts in calculus. I like using this now as a good way to ask students to connect new ideas about rates. This is a minimal introduction only to this learning objective. We will return to this later.*

BIG IDEA 1 Limits	Essential Understandings:	Estimated Time:
	▸ EU 1.1, EU 1.2	12 instructional hours

Guiding Questions:

▸ **How do position and velocity relate?** ▸ **How do we evaluate basic limits?** ▸ **What do limits tell us about the behavior of functions?** ▸ **What is a continuous function?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 2.3F**: Estimate solutions to differential equations.		**Summative Assessment: Basic Graphs of y and y'** Students draw simple graphs, moving from position to velocity and from velocity to position. The notation y and y' is also used. Students are asked to find ordered pairs and sketch graphs generated by Euler's method.	*This summative assessment addresses the following guiding question:* ▸ How do position and velocity relate? *By now, we've done many graphs so that students should be comfortable moving in either direction with relatively simple graphs.*
LO 1.1A(a): Express limits symbolically using correct notation. **LO 1.1A(b)**: Interpret limits expressed symbolically. **LO 1.1B**: Estimate limits of functions. **LO 1.1C**: Determine limits of functions. **LO 1.1D**: Deduce and interpret behavior of functions using limits. **LO 1.2A**: Analyze functions for intervals of continuity or points of discontinuity.	**Print** Hughes-Hallett, chapter 1, sections 7 and 8	**Instructional Activity: Introduction to Limits** I use a sequence of five worksheets to introduce limits. Connections are made with graphs, tables of values, and analytic work. The first examines graphs of piecewise functions with various discontinuities (jump, infinite, and removable) as well as continuous functions to examine when a limit exists and when it does not. This worksheet also introduces the notation for one-sided limits. The second worksheet extends these ideas, using formulas as well as graphs. The third looks at calculator approximations and then introduces algebraic techniques such as factoring and rationalization to find precise answers. The fourth examines limits around asymptotes, both horizontal and vertical. The fifth offers a formal definition of continuity and asks students to justify continuities and discontinuities using limits.	*I use multiple representations and many different types of functions. This should not be done too quickly, as limits will be used and studied later, and a good foundation is essential.* *With enough time, two more worksheets introduce the definition of a limit with delta-epsilon.*

UNIT 1: INTRODUCTION TO RATES AND LIMITS

BIG IDEA 1
Limits

Essential Understandings:
- EU 1.1, EU 1.2

Estimated Time:
12 instructional hours

Guiding Questions:

▸ **How do position and velocity relate?** ▸ **How do we evaluate basic limits?** ▸ **What do limits tell us about the behavior of functions?** ▸ **What is a continuous function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 1.2B: Determine the applicability of important calculus theorems using continuity.	**Print** Hughes-Hallett, chapter 1, section 7	**Instructional Activity: Intermediate Value Theorem (IVT)** Graphs of many types of functions are drawn where the IVT may or may not apply on an interval. A worksheet is used to examine hypotheses and conclusions. Students are asked to explain what the IVT may allow us to conclude in a given context. For example, if I was 23 miles from home two hours ago, and now I'm two miles from home, what, if anything, does the IVT tell me?
All of the learning objectives in this unit are addressed.		**Summative Assessment: Limits** I give a written test on this unit. Given a graph of a position or velocity function, students create possible graphs of the other function. Students calculate and approximate limits of functions given in multiple representations. They also answer questions about continuity and the IVT.

Precise notation should be used and expected on this assessment. Limits should be examined with multiple representations. Some questions about the IVT should examine whether the hypotheses and conclusions hold. This summative assessment addresses the following guiding questions:
- How do we evaluate basic limits?
- What do limits tell us about the behavior of functions?
- What is a continuous function?

Mathematical Practices for AP Calculus in Unit 1

The following activities and techniques in Unit 1 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Introduction to Limits" instructional activity, the definition of continuity is studied and students are asked to determine the continuity of functions at various values of x. Their answers are to be justified using the definition, including one-sided limits. At the end of the unit, students will study the Intermediate Value Theorem. In the "Intermediate Value Theorem" instructional activity students are again asked to answer and justify questions about functions by referring to hypotheses and conclusions.

MPAC 2 — Connecting concepts: The unit begins by connecting position and velocity, and connections are made to the slope of the position function at various points and how the changes in the velocity function affect the concavity of the position function.

MPAC 4 — Connecting multiple representations: This unit first asks student to relate their velocity and position graphs to specific values with Euler's method. Additionally, limits are examined through graphs, tables, and using analytic methods. In several worksheets, students are asked to examine graphs and then to confirm their visual understanding with analytic work.

MPAC 5 — Notational fluency: Notation is particularly important with limits. All worksheets require correct and precise use of notation.

MPAC 6 — Communicating: Students justify answers when calculating limits and with complete sentences when studying the Intermediate Value Theorem.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
13 instructional hours

Guiding Questions:
- How do we measure speed? ▸ How should the derivative be defined? ▸ What does the derivative tell us about a function?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient. **LO 2.1B**: Estimate derivatives.	**Print** Hughes-Hallett, chapter 2, sections 1 and 2 **Software** *Mathematica* or other software	**Instructional Activity: Average Rate of Change** Students calculate the average rate of change over an interval, examining the results as Δx becomes smaller and smaller. This leads to the limit as $\Delta x \to 0$. We continue to sketch graphs of functions with secant and tangent lines. Students use calculators and computer software to draw derivative functions.	*I ask students to make tables of approximations of derivatives over smaller and smaller intervals. The need for a limit evolves naturally. I find it very useful to use computer software to animate the secant line as it approaches the tangent line.*
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Hughes-Hallett, chapter 2, section 3	**Instructional Activity: Given f', Sketch f** As we move toward the definition of a derivative, we simultaneously examine derivative graphs more closely and what they tell us about the original function. I ask what $f' > 0$ and $f' < 0$ tell us about the function f. Then I look for connections between f' and the concavity of the function. Students make a lot of graphs and I ask them to justify their answers. I emphasize the idea that $f'(a) = 0$ does not guarantee an extremum and that $f''(a) = 0$ does not guarantee a point of inflection. We return to $y = x^3$ and $y = x^4$ repeatedly.	
LO 2.2A: Use derivatives to analyze properties of a function.		**Formative Assessment: From f' to f** After practicing graphing and justifications, I give a graph of f' and ask where f is increasing, where it has a maximum, where it has points of inflection, and where f is concave downward, for example. Then I ask students to sketch a possible graph of f.	*I give this as a quiz, done individually, but it is not graded. It is important to give feedback on individual justifications since there are many ways to write answers correctly (and incorrectly!) and students will benefit from individual feedback.*

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
13 instructional hours

Guiding Questions:

- How do we measure speed? ▸ How should the derivative be defined? ▸ What does the derivative tell us about a function?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 2, sections 2 and 3	**Instructional Activity: Finding Derivatives from the Definition** Students begin using limits to calculate derivatives at a specific point and as derivative functions with functions such as $y=2x^2-5x+3$, $y=\frac{3}{x-2}$, and $y=\sqrt{x}$. We use the definition with $y=\lvert x\rvert$ to show that this function does not have a derivative at $x=0$, yet it does have a minimum. Other examples are given. Students write the equations of tangent lines. Connections are made frequently between the derivatives and the graphs.
LO 2.3A: Interpret the meaning of a derivative within a problem.	**Print** Hughes-Hallett, chapter 2, section 4	**Instructional Activity: Interpreting the Derivative** Students are asked to interpret the derivative in a variety of contexts. For example, what is the meaning of the derivative if the primary function is the temperature of a pie in the oven, the cost of a gallon of gas, the population of a certain country, and so on? Students are asked to clearly explain the meaning of a derivative at a specific value.
LO 2.1D: Determine higher order derivatives.	**Print** Hughes-Hallett, chapter 2, section 5	**Instructional Activity: Second Derivative** The second derivative is seen as the derivative of f'. Students connect f'' f'' to changes in f' and to the concavity of f. Other notations such as $\frac{d^2y}{dx^2}$ and $\frac{d^2}{dx^2}(f(x))$ are shown. Students make connections with motion and acceleration. Contexts are given and interpreted in terms of their concavity. For example, consider that costs are still rising, but they are now rising more slowly; what does this say about the signs of the derivatives of the cost function?

I do these problems on many days, only giving a couple each day. In class, we discuss various interpretations and vocabulary to state the meaning explicitly in each context. This is difficult and requires practice over time.

UNIT 2: MEANING OF THE DERIVATIVE

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3

Estimated Time:
13 instructional hours

Guiding Questions:

▸ **How do we measure speed?** ▸ **How should the derivative be defined?** ▸ **What does the derivative tell us about a function?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2B: Recognize the connection between differentiability and continuity.	**Print** Hughes-Hallett, chapter 2, section 6	**Instructional Activity: Differentiable?** The goal here is to clarify and summarize our understanding of the derivative thus far. We've found derivatives and we've seen functions that do not have derivatives at a given value of x. We discuss local linearity and the idea of a smooth function as opposed to a corner or cusp. We prove that a function that is differentiable at a point must also be continuous at that point.
All of the learning objectives in this unit are addressed		**Summative Assessment: Definition and Meaning of a Derivative** I give a written test at this time. I ask students to approximate and calculate simple derivatives by definition, and interpret values of derivatives in context. Students use graphs of a derivative function to determine important features of both f and f''. We move fluidly in both directions between a function and its derivatives.

This summative assessment requires students to have a clear understanding of many fundamental ideas that form the basis of the derivative. It addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 2

The following activities and techniques in Unit 2 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "Finding Derivatives from the Definition" instructional activity asks students to work with the definition. In addition, we visit and prove the theorem that differentiability implies continuity.

MPAC 2 — Connecting concepts: Several instructional activities in this unit, including "Given f', Sketch f" and "Finding Derivatives from the Definition," ask students to make connections between a function and its derivatives, both verbally and graphically.

MPAC 3 — Implementing algebraic/computational processes: Computing derivatives as limits, introduced in the "Finding Derivatives from the Definition" instructional activity requires analytic steps to be done by the students.

MPAC 4 — Connecting multiple representations: Instructional activities such as "Given f', Sketch f" and "Second Derivative" use graphs, analytic functions, and verbal interpretations to help give a deep and broad sense of the meaning of the derivative.

MPAC 5 — Notational fluency: Limits are used repeatedly in this section, along with derivative notation.

MPAC 6 — Communicating: We justify answers verbally when moving from a derivative graph to the graph of the original function in the "Given f', Sketch f" instructional activity. Interpretations of a derivative in context, which students perform in the "Interpreting the Derivative" instructional activity, require careful vocabulary and sentence structure to be unambiguous.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

How do we calculate the derivatives of elementary functions? ▸ How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems? ▸ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 1	**Instructional Activity: Basic Rules** Using the definition of a derivative and making the graphical connection, students note that the derivative of a constant is zero. They see that the derivative of a sum is the sum of the derivatives, and they see that if c is a constant, then $D_x\left(c \cdot f(x)\right)=c \cdot f'(x)$. We also look at the derivative of $y=x^n$. We review some cases that have been done, use a computer algebra system (CAS) to see more examples, note the pattern, and then prove the power rule for $n \in \mathbb{Z}^+$. Students may then calculate derivatives of power functions.	
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 2 **Software** *Mathematica* or a graphing calculator	**Instructional Activity: Exponential Functions** Through a worksheet, students use technology to explore graphs of functions of the form $f(x)=b^x$, initially with $b>1$. They use the definition of the derivative to see that $D_x\left(b^x\right)=b^x \cdot \left(\begin{matrix}\textit{something}\\ \textit{in terms of } b\end{matrix}\right)$. Students look at the ratio $\dfrac{D_x(b^x)}{b^x}$ for various values of b. Plotting points of the form $b, \dfrac{D_x(b^x)}{b^x}$ shows a logarithmic function, and confirming values leads to $f(x)=3\left(\dfrac{1}{4}\right)^x$. Practice is given that asks students to apply this rule.	*This activity is often difficult for students, but I believe it is good for them to struggle with these relationships. Applying the rule to functions with various constants such as $f(x)=3\left(\dfrac{1}{4}\right)^x$ also challenges students. Note that this is not a proof of the derivative formula, but I believe it is good for students to use exponential functions throughout the course.*
LO 2.1C: Calculate derivatives.		**Instructional Activity: Derivatives of Sine and Cosine** I ask students to draw the graphs of sine and cosine. Then I ask them to find or estimate their derivatives at key points and make graphs of the derivative functions. From here, students are convinced of the derivatives of these functions. We do this quickly. Our text examines trigonometric functions a little later. I give a brief introduction here because I want to be able to use the sine and cosine when learning the product and quotient rules, as this offers more variety.	

UNIT 3: CALCULATING DERIVATIVES

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

▸ How do we calculate the derivatives of elementary functions? ▸ How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems? ▸ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 3	**Instructional Activity: Product and Quotient Rules** Rules, practice, proofs, and much more practice. I direct this work in class. I do the proof of the product rule that begins with the definition of the derivative and then a term is added and subtracted from the numerator. Separately, and usually on another day, I do a proof by looking at the change in the area of a rectangle $p(x)=u(x)\cdot v(x)$ with sides $u(x)$ and $v(x)$. For the quotient $q(x)=\frac{u(x)}{v(x)}$, I show the algebra to derive the quotient rule by considering $v(x)\cdot q(x)=u(x)$, though this is not a proof. Students do a lot of practice in class as I move from table to table to help them. Then students show their derivatives on the board for all to check.	
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 4 **Software** *Mathematica* or another CAS	**Instructional Activity: Chain Rule** We explore simple cases of composition, such as $y=\sin(2x)$ and $y=e^{3x}$ to see the effect on the derivative. Then students use a CAS to see the derivatives of a variety of functions. They are asked to find the pattern for the derivative of $y=f(g(x))$. I watch students and give hints as necessary for the pattern. Following that, we do lots of practice, both in class and for homework. As before, I watch students at their seats and I send students to the board.	*I have had good success with this activity, much more so than when I try to explain the chain rule with examples.*
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 5	**Instructional Activity: Trigonometric Derivatives** We now take a very brief detour to look at limits involving trigonometric functions such as $\lim\limits_{x\to\infty}\frac{\sin(\theta)}{\theta}$ and variations. I also show students the Squeeze Theorem. This allows me to prove the rules for the derivatives of sine and cosine using the definition of the derivative. I usually do the sine on the board with help from students. Then students do the work for cosine. Other derivative rules are found by using the quotient rule. With these skills, and in combination with other skills, we also practice problems about graphs, tangent lines, and so on.	

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

▸ How do we calculate the derivatives of elementary functions? ▸ How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems? ▸ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.		**Formative Assessment: Finding Derivatives** I put functions on the board each day for students to differentiate. I walk around checking individual work and giving hints. Students put work on the board to check. In addition, I send students to the board in groups, depending on the size of the class and board space. I call out functions, and students find the derivatives. Students at their seats do the same. Then we rotate. This encourages students to check their skills while being able to gain quick help from others.	*When students are at the board working individually, I can see what each one can do. I can tell a student to check a certain part or to check his or her neighbor's work. I can also suggest that everyone look at different approaches to differentiate a function such as* $f(x)=\sqrt{\dfrac{\sin(3x)}{x^3+5x-2}}$ *using the chain rule first or the quotient or product rule. Depending on what I see, I know what types of functions need more practice.*
LO 2.1C: Calculate derivatives.		**Summative Assessment: Finding and Using Derivatives** I give a test at this time. Much of it is computational. I also add questions about finding the equation of a tangent line at a point, finding second derivatives, and whether functions are increasing or concave down, for example, in keeping previous material alive.	*This summative assessment addresses the following guiding questions:* ▸ How do we calculate the derivatives of elementary functions? ▸ "How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems?

UNIT 3: CALCULATING DERIVATIVES

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

▶ How do we calculate the derivatives of elementary functions? ▶ How do we find derivatives of combinations of functions? ▶ How can we extend the utility of derivatives to solve problems? ▶ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 6	**Instructional Activity: Inverse Functions** Students discuss what they remember about inverse functions in general, such as the composition and graphs of f and its inverse. I ask students to find the value of the derivative of $y = x^3$ at $x = 2$ along with the derivative of $y = x^{1/3}$ at $x = 8$. We make graphs and discuss why these are reciprocals. Then, if $g(x) = f^{-1}(x)$, we differentiate $g\left(f(x)\right) = x$ to obtain the relationship between the derivatives of a function and its inverse. Students practice these ideas with functions such as $f(x) = x^2 + 5x + 3$ at (1, 6) and its inverse. At this point, I can apply this relationship to derive derivative formulas for $y = \sin^{-1}(x)$ and $y = \tan^{-1}(x)$. Then we practice the chain rule with these functions, often at the board as done previously.	*Before working with inverse functions, I give homework to review graphs, domain and range, and special values. I find that inverse function problems are confusing to many, so practice and examining graphs is helpful. I often include the derivative of $y = \sec^{-1}(x)$ as well. I also find it helpful to spend time on the distinction between the domains of $y = \sin^{-1}(x)$ and its derivative.*
LO 2.1C: Calculate derivatives.	**Print** Hughes-Hallett, chapter 3, section 7 **Software** I use *Mathematica*, but other programs can graph implicit equations.	**Instructional Activity: Implicit Functions** I begin with the familiar circle $x^2 + y^2 = 25$, and ask students for value of the derivative at (3, 4). I offer the implicit approach, making comparisons about the results of this method and solving for y and differentiating explicitly. We continue with other examples. I show a lot of graphs on the computer to broaden the students' sense of graphs and to help confirm slopes at specific points. Later, I use worksheets to extend their knowledge to find points where the tangent lines are horizontal or vertical.	*Substituting $f(x)$ for y helps some students see the mechanics for the derivatives more clearly. Computer graphics both entertain and intrigue the students. Many ask questions about how other constants will affect the graph. In particular, the graphics help students understand the process for finding points where the tangent line is horizontal or vertical. I find this isn't obvious to a lot of students.*

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

▸ How do we calculate the derivatives of elementary functions? ▸ How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems? ▸ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.3B: Solve problems involving the slope of a tangent line.	**Print** Hughes-Hallett, chapter 3, section 9	**Instructional Activity: Local Linearity and Theorems** With the ability to write equations of tangent lines, we can now talk more about approximations to functions with the idea of local linearity. We also discuss whether these approximations are too high or too low, depending on concavity. When deriving the definition of a derivative, we discuss local linearity and approximated slopes over small intervals. This is a nice way to reconnect to those early ideas.
LO 1.2B: Determine the applicability of important calculus theorems using continuity. **LO 2.4A:** Apply the Mean Value Theorem to describe the behavior of a function over an interval.	**Print** Hughes-Hallett, chapter 3, section 10	**Instructional Activity: Mean Value Theorem (MVT)** An introductory worksheet has graphs of various functions (continuous and not, differentiable and not) on an interval $[a, b]$. Students connect $(a, f(b))$ and $(b, f(b))$ to show the average rate of change. If possible, they draw tangent lines with the same slope. Then I ask them about features that make such tangent lines possible. This leads to hypotheses and the conclusion of the MVT. We practice the "formula" to find values of c. Students explain the results of the MVT in context: for example, if the temperature was 40 degrees at 7 a.m. and 53 degrees at noon, does the MVT apply? If so, what can you conclude? I ask small groups of students to read and present the theorems that follow from the MVT and the proof of the MVT itself to the class.

With the MVT, it is important for students to see both continuity and differentiability. The mini-presentations are good opportunities for students to read, present, and explain small proofs.

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 2.4

Estimated Time:
22 instructional hours

Guiding Questions:

▸ How do we calculate the derivatives of elementary functions? ▸ How do we find derivatives of combinations of functions? ▸ How can we extend the utility of derivatives to solve problems? ▸ What does the Mean Value Theorem tell us?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 1.2B**: Determine the applicability of important calculus theorems using continuity. **LO 2.4A**: Apply the Mean Value Theorem to describe the behavior of a function over an interval.		**Summative Assessment: Implicit Graphs and the MVT** I give either a test or a long quiz at this time. Students are asked to write equations of tangent lines and points where the tangent line is either horizontal or vertical for implicitly defined curves. As a separate issue, students answer questions about the hypotheses and conclusions of the Mean Value Theorem, and they explain the meaning of this theorem in context.

This summative assessment addresses the following guiding questions:
- How can we extend the utility of derivatives to solve problems?
- What does the Mean Value Theorem tell us?

Mathematical Practices for AP Calculus in Unit 3

The following activities and techniques in Unit 3 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "Local Linearity and Theorems" instructional activity asks students to understand the meaning of the Mean Value Theorem visually and in context, and examine the need for the hypotheses.

MPAC 2 — Connecting concepts: Several instructional activities in this unit, including "Exponential Functions" and "Trigonometric Derivatives," require students to use their precalculus understanding of these functions to make sense of the derivative rules.

MPAC 3 — Implementing algebraic/computational processes: Many instructional activities in this unit, such as "Basic Rules" and "Product and Quotient Rules," begin with a limit definition and then include much practice on using the rules to algebraically compute derivatives.

MPAC 4 — Connecting multiple representations: Several instructional activities in this unit, including "Exponential Functions" and "Product and Quotient Rules" among others, ask students to make connections between the definition and resulting rules using analytic, visual, and numerical approaches.

MPAC 5 — Notational fluency: We use limit notation as well as notation for first and higher-order derivatives repeatedly throughout this unit.

MPAC 6 — Communicating: Students explain the meaning of the Mean Value Theorem both graphically and in various contexts in the "Mean Value Theorem" instructional activity. In addition, students are asked to present theorems and proofs to the class.

UNIT 4: APPLICATIONS OF DERIVATIVES

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 1.1, EU 2.2, EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

▶ How can we optimize a function? ▶ How can we connect rates of changing quantities? ▶ How can we use derivatives to compute limits? ▶ How can we use derivatives to understand parametric equations and graphs?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Hughes-Hallett, chapter 4, sections 1–3	**Instructional Activity: Optimization** Students clarify and review the fundamental connections between derivatives and properties of graphs. Specifically, we review the first derivative test and state the second derivative test formally, though many students already understand this idea. Students practice these ideas on both open and closed intervals. We use these ideas with pure graphs and with given functions in context, and finally, students are expected to create their own models of functions to optimize.
LO 2.2A: Use derivatives to analyze properties of a function.	**Print** Hughes-Hallett, chapter 4, section 4	**Instructional Activity: Families of Functions** Students continue their work on optimization and other graphical properties with functions containing parameters. For example, they find extrema, points of inflection, asymptotes, and so on, of functions such as $f(x)=f(x)=ae^{-x}+bx$ in terms of the given parameters. These problems are often difficult for many students.
LO 2.2A: Use derivatives to analyze properties of a function.	**Software** *Mathematica* or other software with animation capabilities	**Summative Assessment: Family of Functions Project** Students work in pairs on various functions using software to create animations, and use calculus to find extrema and points of inflection based on the parameters. They also make presentations to the class. My list of functions involves trigonometric and exponential functions, powers, and square and cube roots — a wide variety.
LO 2.3D: Solve problems involving rates of change in applied contexts.	**Print** Hughes-Hallett, chapter 4, section 6	**Instructional Activity: Related Rates** Animations are helpful to introduce this concept. Simple calculations at each second of the horizontal and vertical distances for a ladder sliding down a wall help students see that the two rates do indeed vary differently. Students work in groups to put solutions on the board and then explain their method. I like to have students working in groups. I suggest working on this for a couple of days and then doing one or two a day while continuing on to new topics.

I like this project because of the use of technology, presentations, and the text and verbal explanations by students.

This assessment covers the following guiding question:
- How can we optimize a function?

UNIT 4: APPLICATIONS OF DERIVATIVES

BIG IDEA 2
Derivatives

Essential Understandings:
- EU 1.1, EU 2.2, EU 2.3

Estimated Time:
15 instructional hours

Guiding Questions:

- How can we optimize a function? ▸ How can we connect rates of changing quantities? ▸ How can we use derivatives to compute limits? ▸ How can we use derivatives to understand parametric equations and graphs?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 1.1C: Determine limits of functions.	**Print** Hughes-Hallett, chapter 4, section 7	**Instructional Activity: L'Hospital's Rule** Students do an introductory worksheet that examines graphs of two functions with the same x intercept. They find equations of the tangent lines at this intercept, zoom in, and look at the limit of the quotient of the tangent lines. After discussing this more formally, students do many problems in class as I help with the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Some problems are put on the board to check and discuss. More problems involve many different indeterminate forms such as $\infty \cdot 0, 1^{\infty}, \infty - \infty$, for example, and algebraic techniques to deal with these limits.	*It is helpful to push the students' knowledge of functions. Emphasize the importance of understanding how quickly different functions may grow, thus understanding which function dominates. Students should be able to evaluate some limits without L'Hospital's Rule.*
LO 2.3C: Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**Print** Hughes-Hallett, chapter 4, section 8	**Instructional Activity: Parametric Equations** Students use technology (any computer graphics program or website that creates graphs) to graph a variety of equations. In particular, students examine the parametric equations for circles and other conics. For example, we look at $x = 3\cos(t), y = 2\sin(t)$ and $x = 2\sec(t), y = 3\tan(t)$. For more general equations, as done with implicit graphs, students look for points with horizontal and vertical tangent lines. Problems from the text and from worksheets ask for derivatives, tangent lines, important points on the graph, and speed.	
LO 2.3D: Solve problems involving rates of change in applied contexts. **LO 1.1C:** Determine limits of functions. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.		**Summative Assessment: Applications of Derivatives** This test asks students to use derivatives to answer questions about graphs, optimize functions, find how fast a quantity changes at a particular instant, calculate nontrivial limits, and determine information about parametric equations.	*I often divide this test into multiple parts, allowing students more time to think about significant problems. This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 4

The following activities and techniques in Unit 4 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "L'Hospital's Rule" instructional activity asks students to check to see that appropriate conditions hold before applying the rule.

MPAC 2 — Connecting concepts: The "L'Hospital's Rule" instructional activity necessitates the connection with basic graphs from previous courses. The "Parametric Equations" instructional activity connects nicely with implicit equations.

MPAC 3 — Implementing algebraic/computational processes: The "L'Hospital's Rule" instructional activity asks students to do algebraic manipulation and compute derivatives to evaluate limits. All activities in this section require students to find derivatives of a variety of functions.

MPAC 4 — Connecting multiple representations: Most significantly students look at equations, graphs, and animations to make sense of the "Related Rates" instructional activity.

MPAC 5 — Notational fluency: Limit notation is used in the "L'Hospital's Rule" instructional activity and proper notation is key in working with the "Parametric Equations" instructional activity.

MPAC 6 — Communicating: The "Family of Functions Project" summative assessment requires students to explain their understanding of calculus in written form and then verbally in front of the class.

UNIT 5: MEANING OF THE INTEGRAL AND THE FUNDAMENTAL THEOREMS OF CALCULUS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
17 instructional hours

Guiding Questions:

▸ **What is an integral, and how do we define it?** ▸ **What is the meaning of an integral?** ▸ **How do integrals and derivatives connect?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum. **LO 3.2B:** Approximate a definite integral.	**Print** Hughes-Hallett, chapter 5, sections 1 and 2; chapter 7, section 5 **Software** *Mathematica* or other software to animate more and more rectangles under a curve	**Instructional Activity: Measuring Distance** The first worksheet in this sequence, which contains data about velocity and water in a tank, asks students to make upper and lower approximations of the distance traveled and the quantity of water leaked from the tank. This introduces left and right Riemann sums, area, and the (future) idea that integrals measure different types of accumulation. A second worksheet formalizes left, right, and midpoint Riemann sums. The third worksheet requires a calculator program with an increasing number of rectangles. The fourth worksheet introduces trapezoidal approximations. The last two worksheets ask students to examine the role of increasing/decreasing functions and concavity to determine whether approximations are too high or too low.	*Asking students to draw rectangles and make some calculations by hand will help to clarify and deepen their understanding. Animations help to clarify the limiting process as the number of rectangles increases.* *As formative feedback, I collect and check these worksheets.*
LO 3.2B: Approximate a definite integral. **LO 3.2C:** Calculate a definite integral using areas and properties of definite integrals. **LO 3.4A:** Interpret the meaning of a definite integral within a problem. **LO 3.4C:** Apply definite integrals to problems involving motion. **LO 3.4E:** Use the definite integral to solve problems in various contexts.	**Print** Hughes-Hallett, chapter 5, sections 2 and 3	**Instructional Activity: Integral Notation and the Meaning of an Integral** Notation is given for the definite integral. Students calculate integrals using geometrical ideas based on the idea of signed area under a function. Students continue with Riemann sums and trapezoids to approximate integrals given a table of values. In addition, students use technology to evaluate integrals that represent an accumulated total of change over an interval. Examples from the text include distance, displacement, oil leaked out of a tanker, acres depleted through strip mining, pollution over time, and so on.	

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
17 instructional hours

Guiding Questions:

- **What is an integral, and how do we define it?** ▸ **What is the meaning of an integral?** ▸ **How do integrals and derivatives connect?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum. **LO 3.2A(b):** Express the limit of a Riemann sum in integral notation.	**Print** Hughes-Hallett, chapter 5, section 2	**Instructional Activity: Integral Notation and the Definition of an Integral** Overlapping both the content and lessons in the previous instructional activity, I extend the idea of Riemann sums to introduce the definition of the integral as the limit of a Riemann sum. Students use the definition to calculate integrals of linear functions.	
LO 3.3A: Analyze functions defined by an integral. **LO 3.3B(b):** Evaluate definite integrals.		**Instructional Activity: Euler's Method, Again** Euler's method (in the form $y' = f(x)$ only) is reviewed and connected to integrals and accumulation through a sequence of worksheets. These show that steps of Euler's method accumulate the areas of lefthand rectangles under the graph of y'. Students see that this works for positive and negative values of y'. They are led to understand that the antiderivative that Euler's method finds is the same as the integral. They explore the effect of the lower limit of integration, the starting point of the accumulation. Students explore and study functions of the form $A(x) = \int_a^x f(t)dt$ both graphically and analytically.	*This approach is hardly the most direct, but I find that it helps students truly understand the connections between antiderivatives and accumulation in the form of integrals.*
LO 3.2A(a): Interpret the definite integral as the limit of a Riemann sum.		**Summative Assessment: Riemann Sums, Euler, and Accumulation** Students draw rectangles and trapezoids and determine whether approximations are too high or too low. They calculate an integral such as $\int_1^5 (3x-2)dx$ using the definition of an integral and evaluate other integrals using geometry. Other problems ask students to create and calculate integrals representing total change in a quantity.	*This summative assessment addresses the following guiding questions:* ▸ What is an integral, and how do we define it? ▸ What is the meaning of an integral?

UNIT 5: MEANING OF THE INTEGRAL AND THE FUNDAMENTAL THEOREMS OF CALCULUS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
17 instructional hours

Guiding Questions:

▸ **What is an integral, and how do we define it?** ▸ **What is the meaning of an integral?** ▸ **How do integrals and derivatives connect?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals. **LO 3.4B:** Apply definite integrals to problems involving the average value of a function.	**Print** Hughes-Hallett, chapter 5, section 4	**Instructional Activity: Properties of Integrals** Worksheets ask students to explore properties of integrals: linearity, integrating "backward," the geometrical meaning of $\int_a^b \left(f(x)-g(x)\right)dx$, the average value of a function, and $\int_a^b f(x)dx \int_a^c f(x)dx+\int_c^b f(x)dx$. Students use these properties and geometry to evaluate more integrals such as $\int_0^2 \left(3\sqrt{4-x^2}+1\right)dx$ or $\int_{-1}^4 (6-3x)dx$.	*Most of these properties are reasonable for most students, so I let students work them in class. I am careful to check their work and clarify ideas to ensure accuracy.*
LO 3.1A: Recognize antiderivatives of basic functions. **LO 3.3A:** Analyze functions defined by an integral. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Hughes-Hallett, chapter 5, section 3; chapter 6, section 4	**Instructional Activity: The Fundamental Theorems** Students continue to study functions of the form $A(x)=\int_a^x f(t)dt$. Worksheets lead students to evaluation by $F(b)-F(a)$. (Here, antiderivatives are simple power functions.) I do computer animations with a variety of functions to show f and A as x changes. We look at the effect of a. There is a lot of practice graphing A when given a graph of f. Problems ask about extrema and points of inflection. Both forms of the Fundamental Theorems are formally stated. Many more practice problems are done to help students feel very comfortable with these concepts.	*By the time we get here, most students know both FTCs, though not very clearly. This work brings formality and precision to these ideas.*
LO 3.3B(a): Calculate antiderivatives.	**Print** Hughes-Hallett, chapter 6, section 4	**Instructional Activity: Basic Antiderivatives** Having guessed and used a few antiderivatives, we now spend a little more time widening our usage of functions. Students find antiderivatives of power functions, sines and cosines, and the exponential function. We do some work with constants for antiderivatives of functions such as $\sin(2x)$ or e^{4x}.	

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.1, EU 3.2, EU 3.3, EU 3.4

Estimated Time:
17 instructional hours

Guiding Questions:
- **What is an integral, and how do we define it?**
- **What is the meaning of an integral?**
- **How do integrals and derivatives connect?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.2C: Calculate a definite integral using areas and properties of definite integrals. **LO 3.4B:** Apply definite integrals to problems involving the average value of a function. **LO 3.1A:** Recognize antiderivatives of basic functions. **LO 3.3A:** Analyze functions defined by an integral. **LO 3.3B(b):** Evaluate definite integrals. **LO 3.3B(a):** Calculate antiderivatives.		**Summative Assessment: FTCs and Antiderivatives** This test requires students to pull many concepts together. Students set up integrals to find quantities representing total change, noting correct units. Problems include finding basic antiderivatives and evaluating definite integrals by the FTC. Students use the other form of the FTC to find derivatives of functions defined as integrals.

This summative assessment addresses the following guiding questions:
- How do integrals and derivatives connect?
- What is the meaning of an integral?

Mathematical Practices for AP Calculus in Unit 5

The following activities and techniques in Unit 5 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: The "Integral Notation and the Definition of an Integral" instructional activity asks students to work with the definition of an integral and the notation of limits and summations.

MPAC 2 — Connecting concepts: Several instructional activities in this unit, including "Euler's Method, Again" and "The Fundamental Theorems," ask students to make connections between a function and its antiderivative, both analytically and graphically.

MPAC 3 — Implementing algebraic/computational processes: The "Measuring Distance" instructional activity requires students to compute approximations to integrals by specified methods, and the "Basic Antiderivatives" instructional activity requires analytic work.

MPAC 4 — Connecting multiple representations: All activities in this unit make connections between graphs, analytic functions, and/or numerical approximations.

MPAC 5 — Notational fluency: Throughout this unit, students are expected to work with integral notation and functions of the form $A(x)=\int_a^x f(t)\,dt$.

MPAC 6 — Communicating: Students are asked repeatedly in this unit to justify their answers, both with analytic and verbal arguments. For example, the "Integral Notation and the Definition of an Integral" and "The Fundamental Theorems" instructional activities require careful justifications to all questions to ensure full understanding.

UNIT 6: TECHNIQUES OF INTEGRATION

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ How can substitution be used to find antiderivatives? ▸ How does integration by parts help us with integrals of products? ▸ How can partial fractions be used with integrals of rational functions?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.3B(a): Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.	**Print** Hughes-Hallett, chapter 7, section 1	**Instructional Activity: Substitution** I show a few examples using substitution, and then students work together to find more antiderivatives. Students are encouraged to look for composition first. For example, note the composition of the cosine function and $y = x^3 + 1$ in $\int x^2 \cos(x^3 + 1)dx$. Then we let $u = x^3 + 1$, the inside of the composition. Also included are problems with antiderivatives that yield logarithms and inverse trigonometric functions. I put problems on the board, circulate around the class as students work, and then students put solutions on the board to allow other students to check their work. In addition, text problems and worksheets require students to do algebra — including multiplying out, separating one fraction into multiple fractions, completing the square, and long division — before actually finding the antiderivative. Students also learn how to evaluate definite integrals by substituting into the limits of integration.
LO 3.3B(a): Calculate antiderivatives.	**Print** Hughes-Hallett, chapter 7, section 2	**Instructional Activity: Integration by Parts** I use the product rule for derivatives to derive the formula for integration by parts. After a couple of examples, students work together in class while I circulate. As before, some problems are put on the board for all to check their work. We discuss an appropriate order of functions for choosing "u."

I encourage students to have a mental list of things to try — algebraic options and possible substitutions. (Staring is not an effective technique!)

For choosing "u," in 1983, Herbert Kasube wrote an article for the American Mathematical Monthly where he proposed choosing functions according to LIATE:

L: Logarithmic

I: Inverse Trigonometric

A: Algebraic

T: Trigonometric

E: Exponential

UNIT 6: TECHNIQUES OF INTEGRATION

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.3

Estimated Time:
12 instructional hours

Guiding Questions:

▸ **How can substitution be used to find antiderivatives?** ▸ **How does integration by parts help us with integrals of products?** ▸ **How can partial fractions be used with integrals of rational functions?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.3B(a): Calculate antiderivatives.	**Print** Hughes-Hallett, chapter 7, section 4	**Instructional Activity: Partial Fractions** Worksheets lead students through several examples of integrals requiring partial fraction decomposition. We begin with two linear factors and then consider one linear factor and one irreducible quadratic. I am careful to clarify different approaches to finding the appropriate fractions.	
LO 3.3B(a): Calculate antiderivatives.	**Print** Hughes-Hallett, chapter 7, review exercises	**Formative Assessment: Choose Your Method!** Students work in class to choose an appropriate method of finding the antiderivative. We discuss some approaches for making these choices. Mostly, students need experience trying methods, stopping when stuck, and trying another method. Also, it is good for students to see that in many cases multiple approaches will lead to correct antiderivatives.	*I check some student work for each method: this work is particularly important to assess student understanding.*
LO 3.3B(a): Calculate antiderivatives. **LO 3.3B(b):** Evaluate definite integrals.		**Summative Assessment: Finding Antiderivatives** For this test, I give a wide variety of integrals. Students need to think through options and approaches, and they are expected to show persistence if their first attempt is not helpful.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 6

The following activities and techniques in Unit 6 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 2 — Connecting concepts: The "Integration by Parts" instructional activity takes students back to the product rule to derive this method.

MPAC 3 — Implementing algebraic/computational processes: Every activity in this unit expects students to do serious algebraic work. Some of this is based on past mathematics courses, but most of the analytic work is based on calculus concepts.

MPAC 5 — Notational fluency: Throughout this unit, students are expected to work with substitutions and integral notation.

MPAC 6 — Communicating: Within each activity students put work on the board, explain their choices, and clarify their methods to others.

UNIT 7: APPLICATIONS OF INTEGRALS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.4

Estimated Time:
9 instructional hours

Guiding Questions:

- **How can integrals be used to find area?** ▸ **How can integrals be used to find volume?** ▸ **How can integrals be used to find arc length?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Hughes-Hallett, chapter 5, section 4; chapter 8, section 1	**Instructional Activity: Area** Students use integrals with and without technology to find areas of regions. Problems from the text and from worksheets involve functions that are above and below the x-axis and two functions that bound multiple regions. Students also set up integrals using dy as well as dx. For some regions, two integrals may be necessary.	*I encourage students to sketch the region along with a typical rectangle. They should understand that the integrand represents the area of this rectangle.*
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Hughes-Hallett, chapter 8, sections 1 and 2 **Supplies** Honeycomb decorations	**Instructional Activity: Volume** In this activity I introduce solids of revolution. To do this, I use "honeycomb" decorations (available at cheap stores everywhere) that open up to form solids of revolution and I draw a lot of pictures. Physical models are very helpful. Worksheets ask students to find volumes when regions are revolved around a variety of lines. Again, both dy and dx should be used. I generalize these volumes to solids with known cross sections using the foundation $V=\int_a^b A(x)dx$, where $A(x)$ is the area of the cross section. Students should be encouraged to think about a typical "slice." Pictures and models are essential for a strong geometrical understanding.	*I also like to introduce volumes by shells, as I believe this is helpful to force students to think more carefully about the geometry of washers and shells, but this is not necessary.*
LO 3.4C: Apply definite integrals to problems involving motion.	**Print** Hughes-Hallett, chapter 8, section 2	**Instructional Activity: Arc Length** I develop the formula for arc length by starting with the Pythagorean Theorem on a straight line segment that approximates a small part of the curve. With algebra, I create the expression $\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot\Delta x$. I can use the MVT to write $\sqrt{1+\left(f'(x_i)\right)^2}\cdot dx$. With the sum and the limiting process, this becomes an integral. Then I extend this to generate the formula for arc length for parametric equations. Students set up the appropriate integrals and evaluate them by calculator, though we may do a few integrals for special functions by hand. Some problems are taken from the text but most are given by worksheet.	

UNIT 7: APPLICATIONS OF INTEGRALS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.4

Estimated Time:
9 instructional hours

Guiding Questions:

- **How can integrals be used to find area?** ▸ **How can integrals be used to find volume?** ▸ **How can integrals be used to find arc length?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.4C: Apply definite integrals to problems involving motion. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Summative Assessment: Area, Volume, and Arc Length** This test requires students to set up integrals to represent the geometrical quantities in this unit. Students are asked to evaluate a few of these integrals, as this is a required skill.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 7

The following activities and techniques in Unit 7 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Volume" instructional activity we combine the definitions for the volume of a cylinder and for an integral. In the "Arc Length" instructional activity we relate the Pythagorean Theorem to the definition of an integral.

MPAC 2 — Connecting concepts: Students must understand the connection between the integral notation and the concept of summing up rectangles, washers, or slices. This in inherent in the "Area," "Volume," and "Arc Length" instructional activities.

MPAC 3 — Implementing algebraic/computational processes: In each activity, students evaluate some integrals to practice skills.

MPAC 4 — Connecting multiple representations: All activities in this unit make connections between geometric "slices" and an analytic formulation.

MPAC 5 — Notational fluency: Throughout this unit, students are expected to work with integral notation. The limits of integration are particularly important for these computations.

UNIT 8: IMPROPER INTEGRALS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.2

Estimated Time:
5 instructional hours

Guiding Questions:

- **What makes an integral improper?** ▸ **How can an improper integral be evaluated, or how do we know when the integral diverges?** ▸ **How can the comparison and limit comparison tests help us to make conclusions about convergence and divergence?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.	**Print** Hughes-Hallett, chapter 7, section 6	**Instructional Activity: Understanding Improper Integrals** With examples, I define the two issues that make an integral improper: where the function is unbounded within an interval and where the interval is unbounded. I ask students to make approximations (if possible) for several improper integrals. We use technology to inform their guesses, trying to help students build some intuition about convergence and divergence. Limits are used to evaluate these integrals, if possible. I emphasize the idea that each integral is calculated in the usual way where it is well defined, and only afterward is the limit evaluated. Students determine convergence or divergence of many improper integrals, both from the text and from worksheets.	*I am conscious that both integration and limits are difficult for many students, and the combination can be even harder. Correct notation is very important here and should be emphasized.*
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.	**Print** Hughes-Hallett, chapter 7, section 7 **Software** Students use *Mathematica*, but any CAS system is fine	**Instructional Activity: What Converges?** Students examine many examples to see what converges and what does not. They find that $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$. This leads to the comparison test to allow us to make conclusions about the convergence or divergence of many integrals where the functions do not have simple antiderivatives. More problems are done in class and students put solutions on the board.	
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.		**Instructional Activity: Limit Comparison Test** With emphasis on the size of the integrand as $x \to \infty$, students easily accept that if $\int_2^{\infty} \frac{1}{x^2+1}\,dx$ converges by the comparison test, then $\int_2^{\infty} \frac{1}{x^2-1}\,dx$ should also converge because it behaves similarly. This leads to the limit comparison test to determine convergence or divergence. Students do more problems on worksheets.	*I use the limit comparison test here because it allows me to emphasize the fundamental behavior of functions. I also find that learning this now will help when studying infinite series.*

UNIT 8: IMPROPER INTEGRALS

BIG IDEA 3
Integrals

Essential Understandings:
- EU 3.2

Estimated Time:
5 instructional hours

Guiding Questions:

▸ **What makes an integral improper?** ▸ **How can an improper integral be evaluated, or how do we know when the integral diverges?** ▸ **How can the comparison and limit comparison tests help us to make conclusions about convergence and divergence?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.		**Formative Assessment: Justifications** Though both students and I write some justifications on the board, I collect several worksheets to give individual feedback on the writing and justification about convergence or divergence of an improper integral. Students choose the method — integration, comparison test, or limit comparison test — and there are many ways to give correct justification.	*I ask students to show four items: the test name, the inequality or limit, what's known (about the comparison integral), and the conclusion. This structure helps student writing.*
LO 3.2D: (BC) Evaluate an improper integral or show that an improper integral diverges.		**Summative Assessment: Improper Integrals** A test for this unit expects students to find the value of an improper integral or show that the integral diverges by using integration and limits. Other problems ask students to justify convergence or divergence by integrating or by using an appropriate test. There is often a problem that asks students to explain whether an integral is improper or to explain how to deal with an integral with multiple issues.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 8

The following activities and techniques in Unit 8 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: Students must reason with the definition of improper integrals in the "Understanding Improper Integrals" instructional activity. They wrestle with hypotheses and whether to use the comparison or limit comparison tests in the "What Converges?" and "Limit Comparison Test" instructional activities.

MPAC 2 — Connecting concepts: Students must connect their previous work with integration, their knowledge of limits, and their understanding of the size of functions in all of the activities in this unit.

MPAC 3 — Implementing algebraic/computational processes: The "Understanding Improper Integrals" instructional activity requires students to find antiderivatives and evaluate limits.

MPAC 5 — Notational fluency: Students must use correct notation for integration and for limits in the "Understanding Improper Integrals" instructional activity and throughout this unit.

MPAC 6 — Communicating: Students are expected to write clearly in justifying their conclusions in the "What Converges?" and "Limit Comparison Test" instructional activities.

UNIT 9: SEQUENCES AND SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.1, EU 4.2

Estimated Time:
13 instructional hours

Guiding Questions:

▸ How do we define the convergence of sequences and series? ▸ What tests of convergence may be used for infinite series? ▸ What conditions ensure convergence of an alternating series and what is known about error?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.	**Print** Hughes-Hallett, chapter 9, section 1	**Instructional Activity: Sequences** Students will do a wide variety of problems relating to sequences. They will use a formula to find terms, find $\lim_{n\to\infty} a_n$, use terms to find formulas, and graph values to see patterns. Students work with notation for explicit and recursive sequences. Problems include using the notation to find terms and given terms, and find the pattern. Students also graph the functions. Convergence of a sequence is defined, and students calculate limits to justify convergence.	
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.	**Print** Hughes-Hallett, chapter 9, sections 2 and 3	**Instructional Activity: Series** Series are introduced, and geometric series are emphasized. Students compute the sequence of partial sums, allowing us to define the convergence of a series. Graphs of the sequence and the related sequence of partial sums help to clarify concepts, and students find the sum of convergent geometric series and approximate the sums of many other series. We discuss bounded sequences, and we prove the divergence of the harmonic series. We also discuss the nth term test for divergence.	*I find it very helpful to have students calculate terms a_1, a_2, a_3, and a_{10}, for example for several series, along with their partial sums S_1, S_2, S_3, and S_{10}, to clarify these relationships.*
LO 4.1A: Determine whether a series converges or diverges.	**Print** Hughes-Hallett, chapter 9, section 3	**Instructional Activity: Connecting Series and Improper Integrals** A worksheet directs students through the integral test for p-series, making the connection between infinite series and improper integrals. This quickly leads to the comparison test and the limit comparison test to justify convergence or divergence of infinite series. The nth term test is also discussed. Students practice choosing a test of convergence and then clearly writing their justifications.	*Extra time spent on improper integrals will help here. (The time will be necessary in that unit or this one.)*

UNIT 9: SEQUENCES AND SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.1, EU 4.2

Estimated Time:
13 instructional hours

Guiding Questions:

- **How do we define the convergence of sequences and series?** ▸ **What tests of convergence may be used for infinite series?** ▸ **What conditions ensure convergence of an alternating series and what is known about error?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.		**Formative Assessment: Justifying Convergence or Divergence** As with improper integrals, there are many ways to correctly justify convergence or divergence of an infinite series. A variety of examples such as $\sum \frac{k}{2k^3+1}$, $\sum \frac{2+\sin(k)}{k^2}$, and $\sum \frac{1}{k\ln(k)}$ will help students to see the need for different tests.	*Students do problems on worksheets where choosing an appropriate method and showing work clearly are emphasized. Some work is put on the board, and I collect and give individual feedback.*
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.		**Summative Assessment: Sequences and Series So Far** For this test, students need to recognize and use notation relating to both sequences and series. They calculate values of a_k and S_k to demonstrate their understanding and also calculate the sums of geometric series. For a variety of infinite series, they use the integral test, the comparison test, or the limit comparison test to determine and justify convergence.	*This summative assessment addresses the following guiding questions:* ▸ How do we define the convergence of sequences and series? ▸ What tests of convergence may be used for infinite series?
LO 4.1A: Determine whether a series converges or diverges.	**Print** Hughes-Hallett, chapter 9, section 4	**Instructional Activity: Ratio Test** Students are asked to guess the convergence or divergence of various series based on their knowledge of the functions. Then they evaluate $\lim_{n\to\infty} a_n = 0$ for each series to help understand the various pieces of the ratio test. The connection drawn to geometric series with ratio r as the limiting case shows how we move from one term to the next. Students work together and individually to use the ratio test. Another worksheet asks them to choose between using previous tests and the ratio test.	

UNIT 9: SEQUENCES AND SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.1, EU 4.2

Estimated Time:
13 instructional hours

Guiding Questions:

- **How do we define the convergence of sequences and series?**
- **What tests of convergence may be used for infinite series?**
- **What conditions ensure convergence of an alternating series and what is known about error?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.	**Print** Hughes-Hallett, chapter 9, section 4	**Instructional Activity: Alternating Series** Worksheets ask students to use technology to calculate several terms of the sequence of partial sums to see the patterns numerically. Technology helps students to see convergence or divergence graphically. Students approximate sums and then find upper bounds for the errors using the alternating series error bound.
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series.	**Print** Hughes-Hallett, chapter 9, section 4	**Instructional Activity: Absolute and Conditional Convergence** Definitions are given for each of these two types of convergence. An algorithm is given to help students determine the type of convergence for specific series (check that $\lim_{n\to\infty} a_n = 0$, and then check for absolute convergence. If the series is not absolutely convergent but is alternating, check to see if $\lvert a_n \rvert$ decreases to 0). Working with rearrangements helps students to understand the importance of the difference between these concepts. Worksheets are given: Some problems ask students to simply recognize and state the type of convergence or divergence and others ask students to write out their justifications.
LO 4.2B: Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.	**Print** Hughes-Hallett, chapter 9, section 5	**Instructional Activity: Introduction to Power Series** The definition of a power series is given. Students use the ratio test to calculate the interval of convergence. All endpoints are checked individually.

I have students do a lot of calculations to develop a better feel of the numbers for the sum and the error bound. Graphing the sequence and the series helps students enhance their intuition.

UNIT 9: SEQUENCES AND SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.1, EU 4.2

Estimated Time:
13 instructional hours

Guiding Questions:

▸ How do we define the convergence of sequences and series? ▸ What tests of convergence may be used for infinite series? ▸ What conditions ensure convergence of an alternating series and what is known about error?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.1A: Determine whether a series converges or diverges. **LO 4.1A:** Determine whether a series converges or diverges. **LO 4.1B:** Determine or estimate the sum of a series. **LO 4.2B:** Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.		**Summative Assessment: Alternating Series, Types of Convergence, and Power Series** In this test, students are given an alternating series and find S_n (for some n) and find an error bound. Using the ratio test, they determine convergence and intervals of convergence of power series. Students show their understanding of absolute versus conditional convergence.	*This summative assessment addresses all of the guiding questions for the unit.*

Mathematical Practices for AP Calculus in Unit 9

The following activities and techniques in Unit 9 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: Students must reason with the definition of convergence in the "Sequences" and "Series" instructional activities. Additionally, they deal with theorems in the "Ratio Test" and "Alternating Series" instructional activities.

MPAC 2 — Connecting concepts: The "Connecting Series and Improper Integrals" instructional activity directly makes the connections seen in Unit 8 on "Improper Integrals" with infinite series.

MPAC 3 — Implementing algebraic/computational processes: The "Ratio Test" instructional activity requires students to set up and evaluate limits and simplify absolute value inequalities.

MPAC 6 — Communicating: Explanations are expected for the "Justifying Convergence or Divergence" and "Absolute and Conditional Convergence" instructional activities.

UNIT 10: TAYLOR SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.2

Estimated Time:
12 instructional hours

Guiding Questions:

▶ How do we create a polynomial that resembles a function near a given value of x? ▶ How does our understanding of convergence apply to Taylor series? ▶ How do we calculate error bounds when making approximations? ▶ How can we create new series from old ones?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2B**: Write a power series representing a given function.	**Print** Hughes-Hallett, chapter 10, section 1	**Instructional Activity: Sine, Cosine, and Polynomials** Students examine the graphs of $y=\sin x$, $y=x$, and $y=x-\frac{x^3}{6}$ along with their derivatives. At $x=0$, which derivatives (first? second? etc.) are the same and which are different? How do the graphs compare near $x=0$? With answers to these questions, we reverse the process to create a fourth-degree polynomial that resembles the cosine function.	*I direct this discussion. It isn't very long, but it sets the stage for many ideas to follow. It is important to consider the graphs at each step.*
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2C**: Determine the radius and interval of convergence of a power series.		**Instructional Activity: Next Function …** Here I use a set of developmental worksheets to explore more functions. The first worksheet is about $f(x)=\ln(1+x)$. Students construct a cubic function g by setting the function and its first three derivatives equal to the derivatives of the polynomial at $x=0$. They evaluate the function f and their polynomial g at various values of x. They graph both functions and the error function. The pattern of the polynomial is extended to form an infinite series. Students use the ratio test to find the interval of convergence. The alternating series error bound is used to relate the value of x, the number of terms, and the size of the error bound.	*This activity reviews and extends many recent ideas, and it takes some time to tie these together. Still, many students find this intriguing.*
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2B**: Write a power series representing a given function. **LO 4.2C**: Determine the radius and interval of convergence of a power series.	**Print** Hughes-Hallett, chapter 10, sections 1, 2, and 4	**Instructional Activity: Taylor Series** The second worksheet is similar to the first (in the previous activity), but it uses the function $f(x)=\tan^{-1}(x)$, helping students to deepen their understanding of these concepts. The third worksheet looks at the general process of creating a Maclaurin series for a given function and the formula $\frac{f^{(n)}(0)}{n!}x^n$. This is extended to the more general Taylor series centered around $x=a$. The fourth worksheet returns to the cosine and sine functions, extending them to infinite series. The alternating series error bound is used again. Practice is given with more functions from the text.	

UNIT 10: TAYLOR SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.2

Estimated Time:
12 instructional hours

Guiding Questions:

- **How do we create a polynomial that resembles a function near a given value of x?** ▸ **How does our understanding of convergence apply to Taylor series?** ▸ **How do we calculate error bounds when making approximations?** ▸ **How can we create new series from old ones?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.2B: Write a power series representing a given function.		**Formative Assessment: Creating Taylor Series** I often check students' worksheets and ask questions to be sure they are following the ideas behind the development of Taylor series rather than simply using the formula. That is, I want to be sure students understand that the formulas come from setting derivatives of the primary function and the polynomial equal to each other at the center point. In addition, I give a "quiz" (done individually and graded as a quiz, but no grade is recorded) to ask them to create a Taylor series for a given function.	*Asking questions is important here, as it is easy for students to simply rely on the formula and lose track of the origins of the formula. Some work is put on the board, and I collect and give individual feedback.*
LO 4.2B: Write a power series representing a given function.		**Summative Assessment: Creating Taylor Series and Intervals of Convergence** A test on this material requires students to use derivatives to create a new Taylor series. They find the interval of convergence and calculate error bounds for alternating series. This requires a good foundation, but this work remains ongoing.	*This summative assessment addresses the following guiding questions:* ▸ How do we create a polynomial that resembles a function near a given value of x? ▸ How does our understanding of convergence apply to Taylor series? ▸ How do we calculate error bounds when making approximations?
LO 4.2B: Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.	**Print** Hughes-Hallett, chapter 10, section 3	**Instructional Activity: Substitution** The next worksheet requires students to know the Maclaurin series and the intervals of convergence for several basic functions. Substitution is introduced to form new functions from old. For example, substitution is used to create the series for $\cos(3x)$ or for e^{-x^2}. More practice is given from the text.	*Students should know the series for e^x, $\sin(x)$, $\cos(x)$, and $\frac{1}{1-x}$. It's important to consider the interval of convergence when substituting into $\frac{1}{1-x}$.*

UNIT 10: TAYLOR SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.2

Estimated Time:
12 instructional hours

Guiding Questions:

- How do we create a polynomial that resembles a function near a given value of x? ▸ How does our understanding of convergence apply to Taylor series? ▸ How do we calculate error bounds when making approximations? ▸ How can we create new series from old ones?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.2B: Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.	**Print** Hughes-Hallett, chapter 10, section 3	**Instructional Activity: More New from Old — Differentiating and Integrating Series** Another worksheet asks students to differentiate and integrate series term by term. Students should see that this "works." That is, differentiating the series for sine gives the cosine, just as one would hope and expect. Questions about intervals of convergence are included.	
LO 4.2A: Construct and use Taylor polynomials.	**Print** Hughes-Hallett, chapter 10, section 4	**Instructional Activity: Lagrange Error** I give two forms of the Lagrange error bound: one that uses the upper bound of the $(n+1)st$ derivative and one that uses this derivative at an unknown c in the interval between the center of the Taylor series and the value of x. Worksheet problems require students to use the error bound for e^x and other functions. I create information about general functions to allow students to find upper bounds for the errors using the structure of the Lagrange error bound. I connect the alternating series error bound to the Lagrange error bound for the sine function. Another worksheet begins with $f^{(3)}(x) < K_3$, a constant, and develops the form of the error bound in this case.	*This material is particularly difficult for many students, but with enough practice and repetition of the general form, students can make sense of it. I often lead students through the developmental worksheet, but they can follow the steps and believe that it's not all mysterious.*
LO 4.2B: Write a power series representing a given function.	**Print** Hughes-Hallett, chapter 10, section 3	**Instructional Activity: Using Series to Find Limits and Integrals** An additional worksheet asks students to use series to evaluate limits. By substituting a few terms of the series into the expression for the function in the limit, it is often very easy to evaluate the limit. Similarly, this may be done for integrals such as $\int_0^{0.7} \cos(x^2)dx$. The integration itself becomes very simple, allowing them to deal with integrals that do not have an antiderivative formula. The alternating series error bound is used for integral problems.	

UNIT 10: TAYLOR SERIES

BIG IDEA 4
Series (BC)

Essential Understandings:
- EU 4.2

Estimated Time:
12 instructional hours

Guiding Questions:

- **How do we create a polynomial that resembles a function near a given value of x?** - **How does our understanding of convergence apply to Taylor series?** - **How do we calculate error bounds when making approximations?** - **How can we create new series from old ones?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 4.2B: Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series. **LO 4.2A:** Construct and use Taylor polynomials.		**Formative Assessment: Creating Series** Over the course of many days, I've given students additional problems that require them to create new series and to find upper bounds for the error. I've checked many of these problems done for homework. Other problems are done in class and solutions are put on the board for all to see and check.	*Students often need a lot of practice with these ideas, and especially with error. It is easy to get answers that are close but incorrect, so it is important to check work carefully before a graded assessment.*
LO 4.2A: Construct and use Taylor polynomials. **LO 4.2B:** Write a power series representing a given function. **LO 4.2C:** Determine the radius and interval of convergence of a power series.		**Summative Assessment: Creating and Using Taylor Series and Error** The unit test at this time pulls together many different conceptual ideas. Problems include asking students to create a Taylor series from scratch and to create others by manipulation — substitution, differentiation, or integration. Other problems ask for different approximations of functions and integrals. Error bounds are analyzed using the alternating series or Lagrange error bound.	*This summative assessment addresses the following guiding questions:* - How can we create new series from old ones? - How do we calculate error bounds when making approximations?

Mathematical Practices for AP Calculus in Unit 10

The following activities and techniques in Unit 10 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: Students must reason with the conditions that require the alternating series error bound as well as those that necessitate the Lagrange error bound.

MPAC 2 — Connecting concepts: In the "Taylor Series" instructional activity students need to connect the derivatives of polynomials to the current function in question, and they must use series to evaluate limits where they previously used L'Hospital's Rule in the "Using Series to Find Limits and Integrals" instructional activity.

MPAC 3 — Implementing algebraic/computational processes: The "Creating Taylor Series and Intervals of Convergence" instructional activity requires students to use the ratio test and inequalities to find the intervals of convergence. In the "Substitution" and "More New from Old — Differentiating and Integrating Series" instructional activities students do analytic work to create new series.

MPAC 4 — Connecting multiple representations: In all activities, students examine a function with its Taylor polynomial and the graphs of both. They examine the numerical value of error as a distance between the real value and the approximation.

MPAC 5 — Notational fluency: Good, precise notation is required to work with the "Lagrange Error" instructional activity.

MPAC 6 — Communicating: Students must justify their work with error clearly in the "Lagrange Error" instructional activity and with alternating series.

UNIT 11: POLAR GRAPHS AND VECTORS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.4

Estimated Time:
8 instructional hours

Guiding Questions:

▸ How do we write equations of tangent lines to polar curves? ▸ How do integrals help us find areas bounded by polar curves? ▸ What are the connections among vectors, motion, and parametric curves?

Learning Objectives	Materials	Instructional Activities and Assessments
Precalculus review that does not address specific AP Calculus learning objectives.	**Print** Hughes-Hallett, chapter 8, section 3	**Instructional Activity: Review of Polar Graphs** Worksheets lead students through a quick review of plotting points in polar coordinates, nonuniqueness of polar coordinates, and basic graphs of circles, limaçons, and rose leaves. For us this is review, but it is important to do as students haven't seen this material in over a year.
LO 2.1C: Calculate derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function.	**Print** Hughes-Hallett, chapter 8, section 3	**Instructional Activity: Derivatives and Tangent Lines** First, it is necessary to understand the meanings of $\frac{dr}{d\theta}$ and $\frac{dy}{dx}$ in the context of polar graphs. For a tangent line $\frac{dy}{d\theta}$ is needed, but to do this it is necessary to use the parametric form with $x = r\cos\theta$, $y = r\sin\theta$. Students practice finding equations of tangent lines and they look for points with horizontal or vertical tangent lines.
LO 3.4D: Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.	**Print** Hughes-Hallett, chapter 8, section 3	**Instructional Activity: Area with Polar Graphs** We review the area of a sector of a circle, and then use the concept of the integral to take the sum of sectors and then the limit as the number of sectors increases. We use antiderivatives to calculate the area of a limaçon such as $r = 2 + \cos\theta$ with $0 < \theta \leq 2\pi$. Students do problems from the text and worksheets to find the areas of many regions. In particular, attention is given to the limits of integration and regions bounded by two curves.

Note that many of the derivatives and evaluations become messy very quickly. It's a good opportunity for calculator use.

Limits of integration are particularly difficult for students with polar graphs. We do use antiderivatives on some problems and use the calculator on others.

UNIT 11: POLAR GRAPHS AND VECTORS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.4

Estimated Time:
8 instructional hours

Guiding Questions:

▶ **How do we write equations of tangent lines to polar curves?** ▶ **How do integrals help us find areas bounded by polar curves?** ▶ **What are the connections among vectors, motion, and parametric curves?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.1C: Calculate derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Formative Assessment: Polar Functions** I send students to the board and call out functions such as $r = 3\cos(2\theta)$ and $r = 2 - 4\sin(\theta)$. Students draw graphs of basic functions, allowing all of us to see how much students know and who needs additional help. Students will benefit from a familiarity with basic polar graphs, as this helps them to deal with areas and tangent lines.	*I often collect homework to give feedback, and I ask students to do problems individually in class.*
LO 2.1C: Calculate derivatives. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion.	**Print** Hughes-Hallett, appendix D	**Instructional Activity: Vectors** I introduce the concept of a vector-valued function and its derivative. Problems about motion allow us to connect earlier work with parametric curves to questions about velocity, speed, acceleration, and distance traveled. We do some simple modeling problems with projectiles, given an initial position and initial velocity at a given angle.	*Much of this work is about vocabulary and notation. It is a good opportunity to review and extend earlier work with parametric curves.*

UNIT 11: POLAR GRAPHS AND VECTORS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.1, EU 2.2, EU 2.3, EU 3.4

Estimated Time:
8 instructional hours

Guiding Questions:

▶ How do we write equations of tangent lines to polar curves? ▶ How do integrals help us find areas bounded by polar curves? ▶ What are the connections among vectors, motion, and parametric curves?

Learning Objectives	Materials	Instructional Activities and Assessments
LO 2.1C: Calculate derivatives. **LO 2.2A:** Use derivatives to analyze properties of a function. **LO 2.3C:** Solve problems involving related rates, optimization, rectilinear motion, **(BC)** and planar motion. **LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.		**Summative Assessment: Polar and Vector Functions** This test requires students to graph curves and to calculate and explain derivatives. They write equations of tangent lines and set up and evaluate integrals to find areas bounded by polar curves. Vector problems ask students to calculate speed and distance traveled as well as to set up simple projectile motion models.

This summative assessment addresses all of the guiding questions for the unit.

Mathematical Practices for AP Calculus in Unit 11

The following activities and techniques in Unit 11 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 1 — Reasoning with definitions and theorems: In the "Derivatives and Tangent Lines" instructional activity we return to the definition of a derivative to understand $\frac{dy}{d\theta}$ and $\frac{dy}{dx}$ in the context of polar functions. In the "Area with Polar Graphs" instructional activity we rely on the definition of a sector from geometry as well as the definition of an integral.

MPAC 2 — Connecting concepts: In the "Vectors" instructional activity students need to connect their understanding of parametric curves to the new vocabulary and notation of vector-valued functions.

MPAC 3 — Implementing algebraic/computational processes: In the "Derivatives and Tangent Lines" and "Area with Polar Graphs" instructional activities students calculate derivatives, solve equations involving trigonometric functions, and find antiderivatives.

MPAC 4 — Connecting multiple representations: In all activities, students connect their graphs to their analytic work with tangent lines, area, and motion.

MPAC 5 — Notational fluency: The "Vectors" instructional activity requires students to use the notation of vectors with their previous understanding of derivatives and motion.

UNIT 12: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.3, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

▸ What is a differential equation and what is a solution? ▸ How do slope fields and Euler's method help us understand DEs? ▸ How can models help us understand rates of growth?

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 2.3E: Verify solutions to differential equations.	**Print** Hughes-Hallett, chapter 11, section 1	**Instructional Activity: What's a Differential Equation (DE)?** I lead a class discussion to begin this topic, starting with a simple example such as $y' = 4x + 3$ so that students can find the general solution as a family of familiar curves or a specific solution if given a specific point. Then I remind students of their previous work with motion, where they started with $a(t) = -9.8m/s^2$ and initial conditions for velocity and height. We move to less familiar examples where y' is a function of y. Students do problems to determine whether or not a given function is a solution.	*It's important to give students some time to become comfortable with the look of these equations and what it means to have a solution. It is good to include the idea of modeling if possible at this stage.*
LO 2.3F: Estimate solutions to differential equations.	**Print** Hughes-Hallett, chapter 11, section 2	**Instructional Activity: Slope Fields** I lead students through the idea of slope fields with a couple of simple examples, and then ask them to create one. When $\frac{dy}{dt} = f(y)$ we note patterns where $y' = 0$, indicating equilibrium solutions, and we look at patterns of slope fields resulting from DEs that are functions of one variable only ($y' = f(t)$ or $y' = f(y)$). Students do matching problems, create more slope fields, and draw typical solution curves.	*It's important for students to understand that slope fields can help one understand DEs and long-term behavior based on various initial conditions, but they do not guarantee anything! (They simply don't give enough specific information.)*
LO 2.3F: Estimate solutions to differential equations.	**Print** Hughes-Hallett, chapter 11, section 3	**Instructional Activity: Euler's Method** Start at one point, take a step along the tangent line, check the rate again, follow the new slope for another step, repeat. The basic idea, $\text{new } y = \text{old } y + \Delta y$, is fairly easy for students as a method for approximating solutions to DEs. Still, there are lots of numbers floating around, so students need practice to put the correct number in the correct location. We do some problems by hand, with relatively large step sizes to understand the process, and others with technology to see the sequence of graphical approximations that begin to approach the actual solution.	*Students need to show the process and their steps clearly, either with a labeled chart or with equations. (A bunch of numbers on the page is not sufficient, even if there is a box around an answer!)*

UNIT 12: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.3, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

- **What is a differential equation and what is a solution?**
- **How do slope fields and Euler's method help us understand DEs?**
- **How can models help us understand rates of growth?**

Learning Objectives	Materials	Instructional Activities and Assessments	
LO 3.5A: Analyze differential equations to obtain general and specific solutions.	**Print** Hughes-Hallett, chapter 11, section 4	**Instructional Activity: Separation of Variables** Students learn to separate the variables and then find antiderivatives of both sides of the equation. I show several examples to help students with the algebra of dealing with the arbitrary constant and solving for the dependent variable. Students do many problems with different types of functions. Students find general solutions for some problems, and they use initial conditions to find specific solutions for other problems.	
LO 2.3F: Estimate solutions to differential equations. **LO 3.5A:** Analyze differential equations to obtain general and specific solutions.		**Formative Assessment: Multiple Views** Students begin with a differential equation, draw possible solution curves on a slope field, do a few steps with Euler's method, and solve the DE given a specific point.	*Feedback to students is important to see that they can make the connections between multiple representations.*
LO 2.3E: Verify solutions to differential equations. **LO 2.3F:** Estimate solutions to differential equations. **LO 3.5A:** Analyze differential equations to obtain general and specific solutions.		**Summative Assessment: Multiple Views** On this test, students show that they understand what a DE is and that they can verify a solution. They create a small slope field or match equations and slope fields. They compute a few steps of Euler's method, and they solve differential equations using separation of variables.	*This summative assessment addresses the following guiding questions:* - What is a differential equation and what is a solution? - How do slope fields and Euler's method help us understand DEs?

UNIT 12: DIFFERENTIAL EQUATIONS

BIG IDEA 2
Derivatives
BIG IDEA 3
Integrals

Essential Understandings:
- EU 2.3, EU 3.5

Estimated Time:
10 instructional hours

Guiding Questions:

- **What is a differential equation and what is a solution?**
- **How do slope fields and Euler's method help us understand DEs?**
- **How can models help us understand rates of growth?**

Learning Objectives	Materials	Instructional Activities and Assessments
LO 3.5B: Interpret, create, and solve differential equations from problems in context.	**Print** Hughes-Hallett, chapter 11, sections 5–7	**Instructional Activity: Modeling** Students use worksheets to examine several models. For each model students look at the slope field, compute a few steps of Euler's method, and solve the DE, usually with an initial condition. Students are asked how the real solution compares with the Euler's method approximation. It is important for students to see the relationships among the various approaches, and it also gives more practice with each approach. Students consider exponential growth, Newton's law of cooling (where we often collect data), and logistic population growth. If there is time, we also look at drugs in the bloodstream, salt mixtures, and other models. The connections are important: they allow students to understand the utility of each approach and to see the significance of DEs in general.
LO 2.3E: Verify solutions to differential equations. **LO 2.3F**: Estimate solutions to differential equations. **LO 3.5A**: Analyze differential equations to obtain general and specific solutions. **LO 3.5B**: Interpret, create, and solve differential equations from problems in context.		**Summative Assessment: Differential Equations** This unit test requires students to show all approaches to DEs: graphical (slope fields and solution curves), numerical (Euler's method), analytical (verifying solutions and finding solutions by separation of variables), and verbal (explanations of the DE, long-term behavior, etc.). The emphasis at this point is on making the connections among models, approximate and exact solutions, and their meanings.

This summative assessment addresses the following guiding questions:
- How do slope fields and Euler's Method help us understand DEs?
- How can models help us understand rates of growth?

Mathematical Practices for AP Calculus in Unit 12

The following activities and techniques in Unit 12 help students learn to apply the Mathematical Practices for AP Calculus (MPACs):

MPAC 2 — Connecting concepts: The “Modeling” instructional activity requires students to pull together their understanding of multiple approaches to understanding DEs, their models, and their solutions.

MPAC 3 — Implementing algebraic/computational processes: The “Euler’s Method” and “Separation of Variables” instructional activities ask students to do multiple steps when attempting to find solutions (approximate or real) to DEs.

MPAC 4 — Connecting multiple representations: In particular, the “Modeling” instructional activity asks students to bring together the meanings of the various approaches and representations used to study DEs.

MPAC 5 — Notational fluency: Notation is key when solving DEs in the “Separation of Variables” instructional activity.

MPAC 6 — Communicating: Students must show and explain the meaning of their work in the “Modeling” instructional activity.

Resources

General Resources

Hughes-Hallett, Deborah, et al. *Calculus: Single Variable*. 6th ed. Hoboken, NJ: Wiley, 2013.

Mathematica. Wolfram Research, Inc. **http://www.wolfram.com/mathematica/.**

Supplementary Resources

"Wolfram Demonstrations Project." **http://demonstrations.wolfram.com**. (Animations and demonstrations for all to use with a free player program.)